The Three-Vortex Problem

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Helmholtz's 1858 paper

Über Integrale der hydro-dynamischen Gleichungen, welche den Wirbelbewegungen entsprechen. Journal für reine und angewandte Mathematik 55, 25-55.

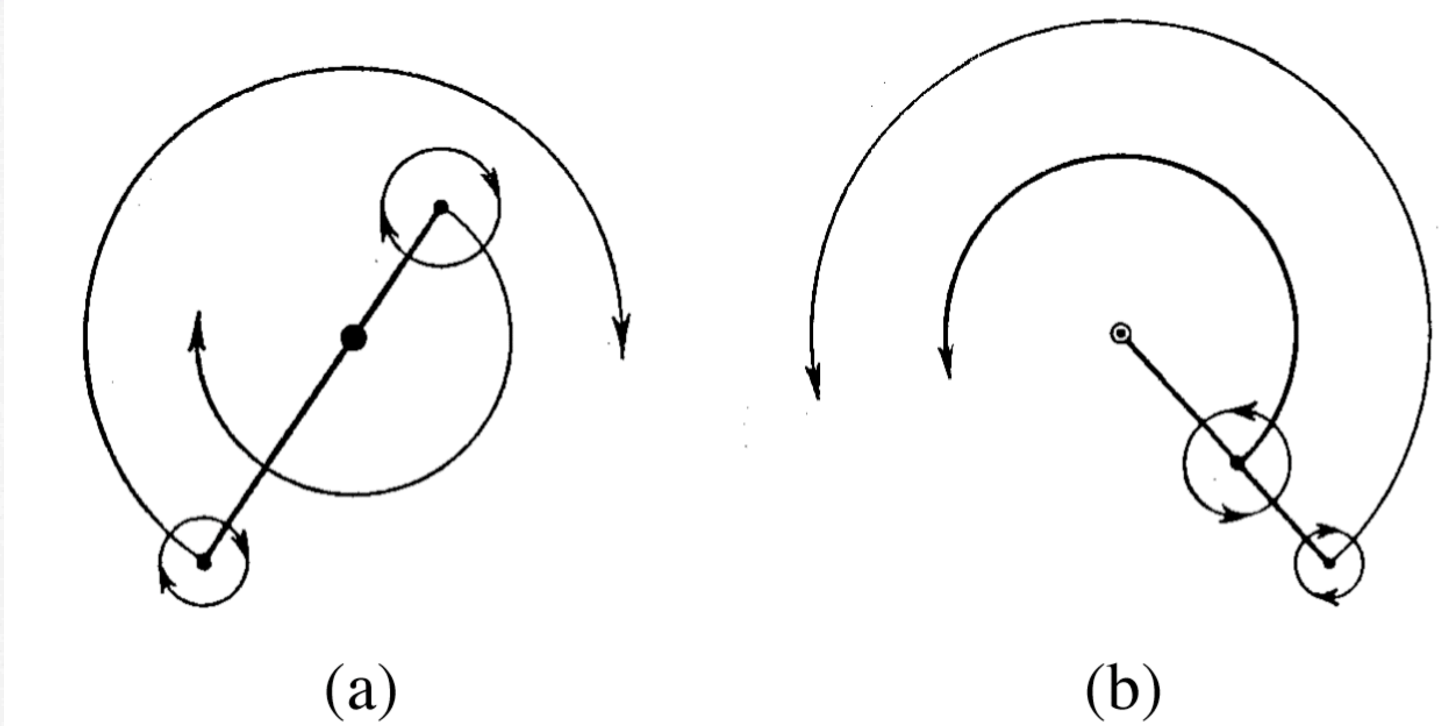
Positions: $z_\alpha = x_\alpha + iy_\alpha$; $\alpha = 1, \ldots, N$

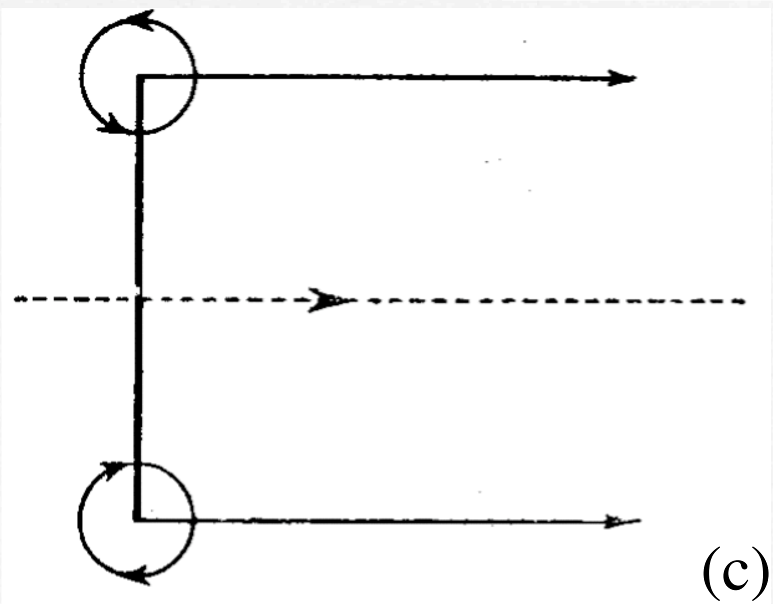
Circulations or "strengths": Γ_α

$$\dot{z}^*_\alpha = \frac{1}{2\pi i} \sum_{\beta=1}^{N}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}$$

Asterisk means complex conjugation; prime on summation means $\beta \neq \alpha$

Two-vortex motion: vortices (a) of the same sign; (b) of opposite sign; and (c) a vortex pair.





Diagrams from
N. E. Joukovskii, [Helmholtz's] works on mechanics, *Uchen. Zap. Imp. Moskov. Univ.* **9** (1892) 37–52 (in Russian)

Kirchhoff's 1877 lectures

Point vortex equations can be written in Hamiltonian form:

$$\Gamma_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha} \qquad \Gamma_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}$$

where H is the Hamiltonian:

$$H = -\frac{1}{4\pi} \sum_{\alpha,\beta=1}^{N}{}' \, \Gamma_\alpha \Gamma_\beta \, log\left| z_\alpha - z_\beta \right|$$

Integrals of the motion

From translational invariance: $X + iY = \sum_{\alpha=1}^{N} \Gamma_\alpha z_\alpha$

From rotational invariance: $I = \sum_{\alpha=1}^{N} \Gamma_\alpha \left| z_\alpha \right|^2$

From invariance to time translation: H itself

Poisson bracket algebra

Definition of the Poisson bracket:

$$[f, g] = \sum_{\alpha=1}^{N} \frac{1}{\Gamma_\alpha}\left(\frac{\partial f}{\partial x_\alpha}\frac{\partial g}{\partial y_\alpha} - \frac{\partial f}{\partial y_\alpha}\frac{\partial g}{\partial x_\alpha}\right)$$

Fundamental Poisson brackets:

$$[z_\alpha, z_\beta] = 0 \qquad [z_\alpha, z_\beta^*] = -\frac{2i\delta_{\alpha\beta}}{\Gamma_\alpha}$$

Key results: $[X, Y] = \sum_{\alpha=1}^{N} \Gamma_\alpha$, $[X, I] = 2Y$, $[Y, I] = -2X$

Thus, $[X^2 + Y^2, I] = 2X[X, I] + 2Y[Y, I] = 0$

(E. Laura, 1904)

Gröbli (1877): Explicit reduction to quadratures (arbitrary circulations)

Poincaré (1893): Always three integrals in involution: H, I and $X^2 + Y^2$. Thus, three-vortex problem (on unbounded plane) is integrable for arbitrary set of vortex strengths.

Walter Gröbli (1852-1903)

Synge (1949): Geometrical interpretation of Gröbli's solutions

Zweites Semester. Von 24. April 1876 bis 15. August 1876

Vorlesungen.	Vermerk des Quästors betreffend das Honorar.	Nummer des Platzes im Auditorio.	Eigenhändige Einzeichnung des Docenten.	Datum der Anmeldung.	Abgemeldet bei dem Docenten.	Datum der Abmeldung.
1. Prof. G. Kirchhoff Theorie d. Wärme.	1 bezahlt	32	G. Kirchhoff	6/5 76.	G. Kirchhoff	26/7 76.
2. Prof. Helmholtz Electrodynamik	2	24	Helmholtz	4/5 76	Helmholtz	24/7 76
3. Prof. Helmholtz Die log. Principien d. Erfahrungswiss.	3 publice	38				
4. Prof. Weierstrass Ergänzungen zur Theorie d. Abel'schen F.	4 ~~[illegible]~~		Weierstraß	3. 5 76	Weierstraß	24/7 76
5. Prof. Kummer Principien der Wahrscheinlichkeiten	5 4/5 76	85	Kummer	17/5 76.	Kummer	26/7 76
6. Prof. Weierstrass Theorie d. anal. Functionen	6 bezahlt ~~[illegible]~~ 20/7 76	62	Weierstraß	22. 5 76	Weierstraß	24/7 76.

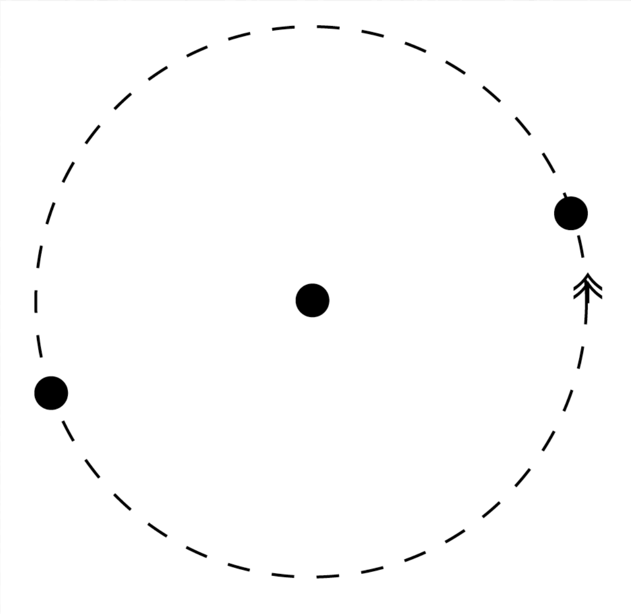
Summary of first 120 years

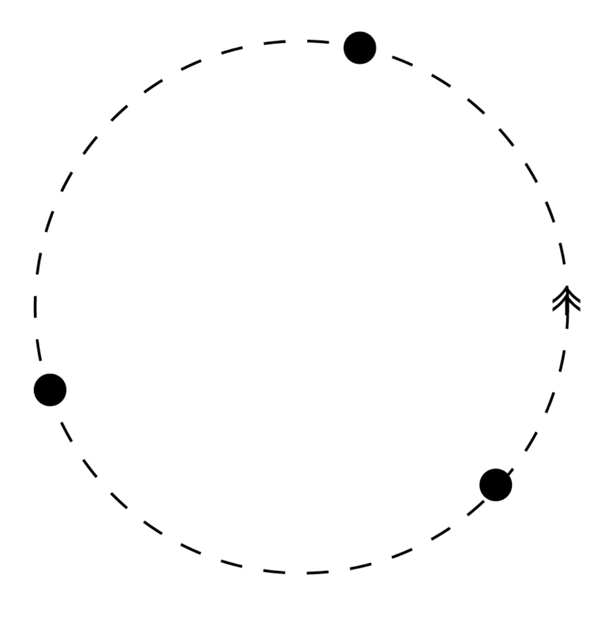
- 1858: Helmholtz's paper on vortex dynamics
- 1877: Kirchhoff's lectures on mechanics
- 1877: Gröbli's thesis on three-vortex problem
- 1878: Simple solutions for identical vortices by Kelvin
- 1893: "Théorie des Tourbillons" by Poincaré
- 1949: Synge's paper on three-vortex problem
- 1975: Novikov's paper on the three-vortex problem
- 1979: Aref's paper on the three-vortex problem

SIR W. THOMSON (LORD KELVIN)

Floating magnets [illustrating vortex systems]

Nature, Vol. XVIII (1878) 13-14





William Thomson

1824-1907

$$\frac{ds_1^2}{dt} = \frac{\Gamma_1}{\pi} 2\Delta \frac{s_3^2 - s_2^2}{s_2^2 s_3^2},$$
$$\frac{ds_2^2}{dt} = \frac{\Gamma_2}{\pi} 2\Delta \frac{s_1^2 - s_3^2}{s_3^2 s_1^2},$$
$$\frac{ds_3^2}{dt} = \frac{\Gamma_3}{\pi} 2\Delta \frac{s_2^2 - s_1^2}{s_1^2 s_2^2},$$

$$16\Delta^2 = 2s_1^2 s_2^2 + 2s_2^2 s_3^2 + 2s_3^2 s_1^2 - s_1^4 - s_2^4 - s_3^4$$

Constants of the motion:

sign of

$$\frac{s_1^2}{\Gamma_1} + \frac{s_2^2}{\Gamma_2} + \frac{s_3^2}{\Gamma_3}$$

Geometric representation:

$$b_1 = \sigma \frac{s_1^2}{\Gamma_1}, \; b_2 = \sigma \frac{s_2^2}{\Gamma_2}, \; b_3 = \sigma \frac{s_3^2}{\Gamma_3}$$

$$\frac{\log s_1}{\Gamma_1} + \frac{\log s_2}{\Gamma_2} + \frac{\log s_3}{\Gamma_3}$$

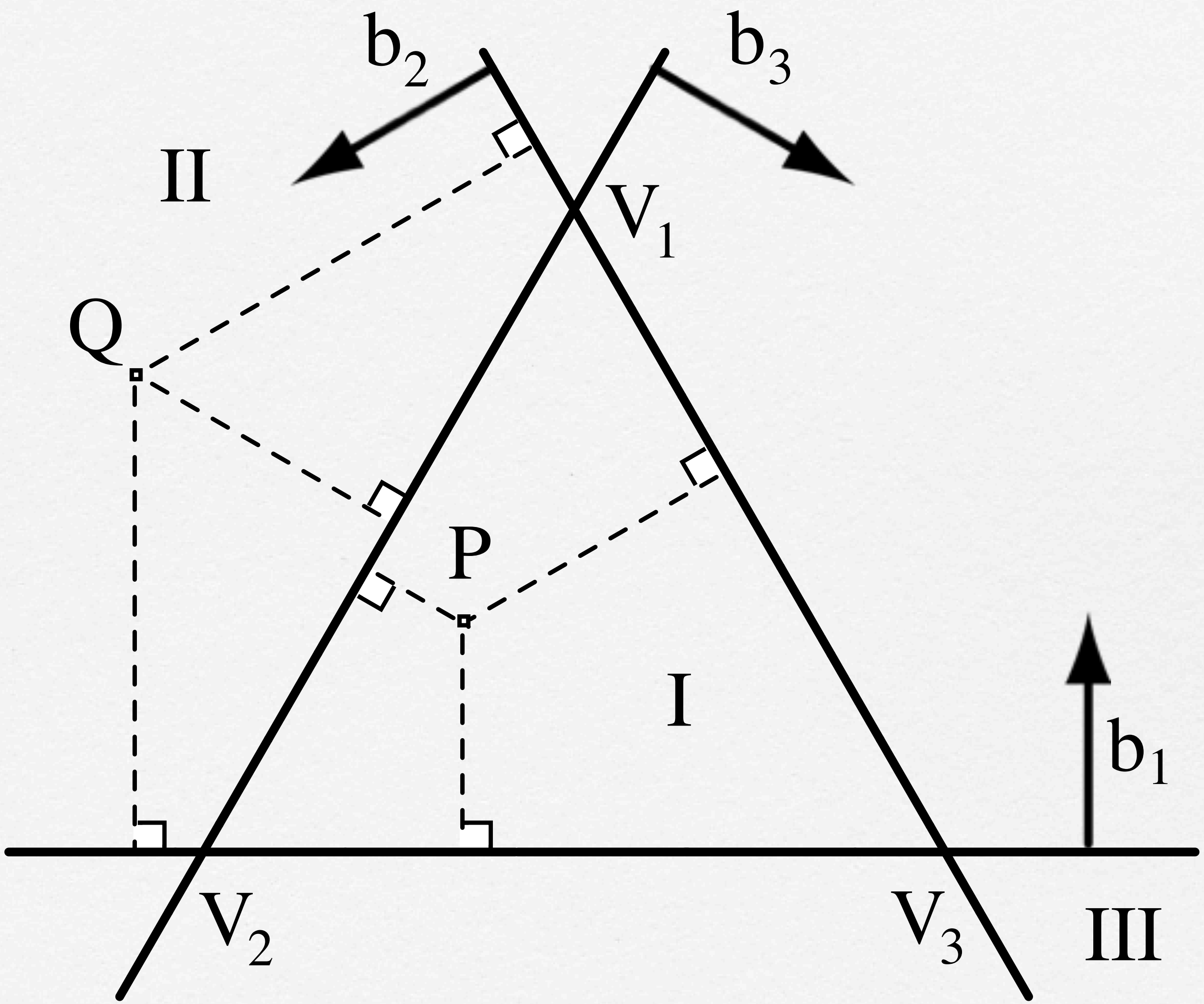
$$b_1 + b_2 + b_3 = 1$$

$$|b_1|^{1/\Gamma_1} |b_2|^{1/\Gamma_2} |b_3|^{1/\Gamma_3} = \frac{1}{\Theta}$$

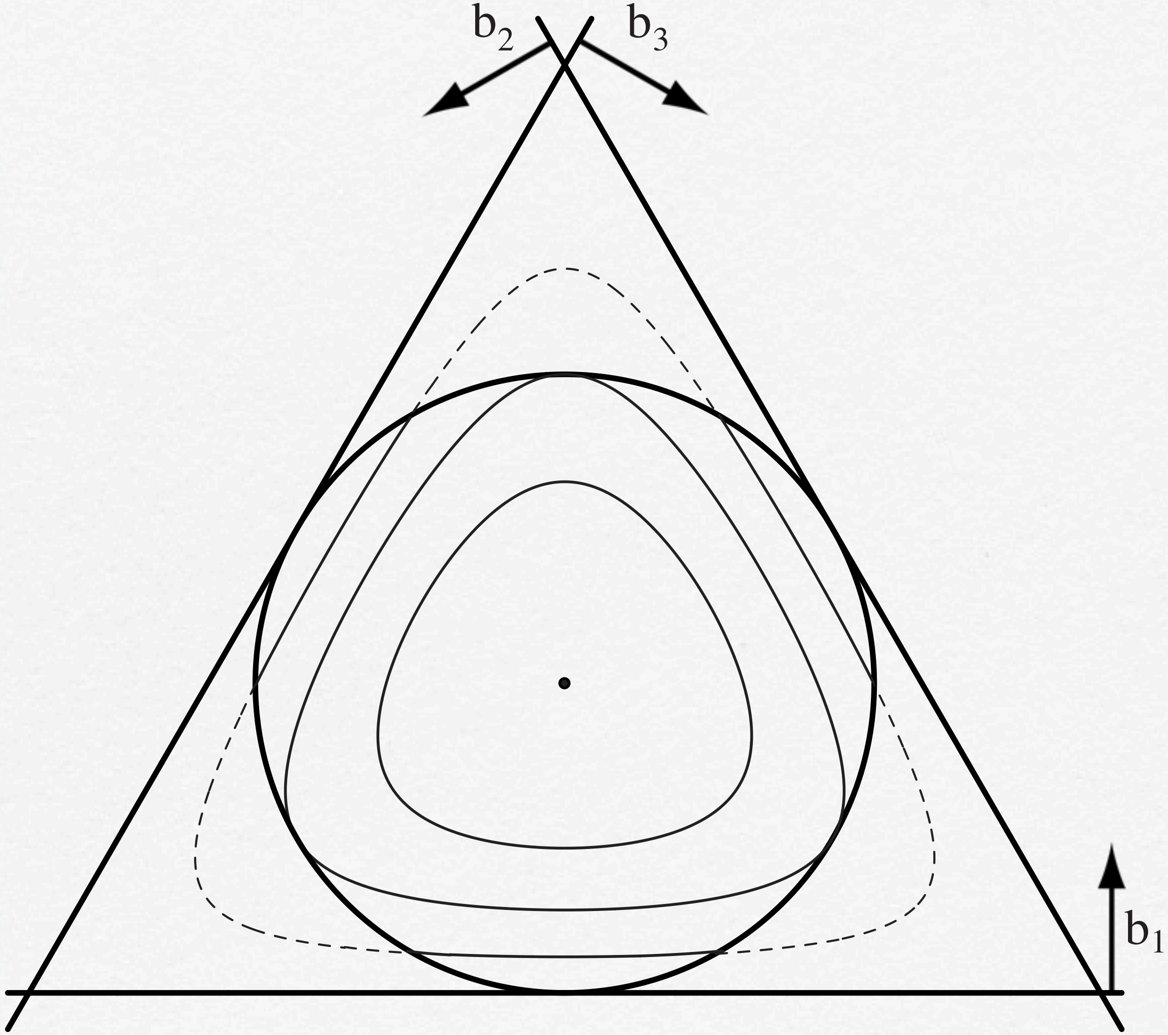
Triangle inequalities:

$$2(\Gamma_1\Gamma_2 b_1 b_2 + \Gamma_2\Gamma_3 b_2 b_3 + \Gamma_3\Gamma_1 b_3 b_1) \geq (\Gamma_1 b_1)^2 + (\Gamma_2 b_2)^2 + (\Gamma_3 b_3)^2$$

Viviani's theorem



$$2(\Gamma_1\Gamma_2 b_1 b_2 + \Gamma_2\Gamma_3 b_2 b_3 + \Gamma_3\Gamma_1 b_3 b_1) \geq (\Gamma_1 b_1)^2 + (\Gamma_2 b_2)^2 + (\Gamma_3 b_3)^2.$$



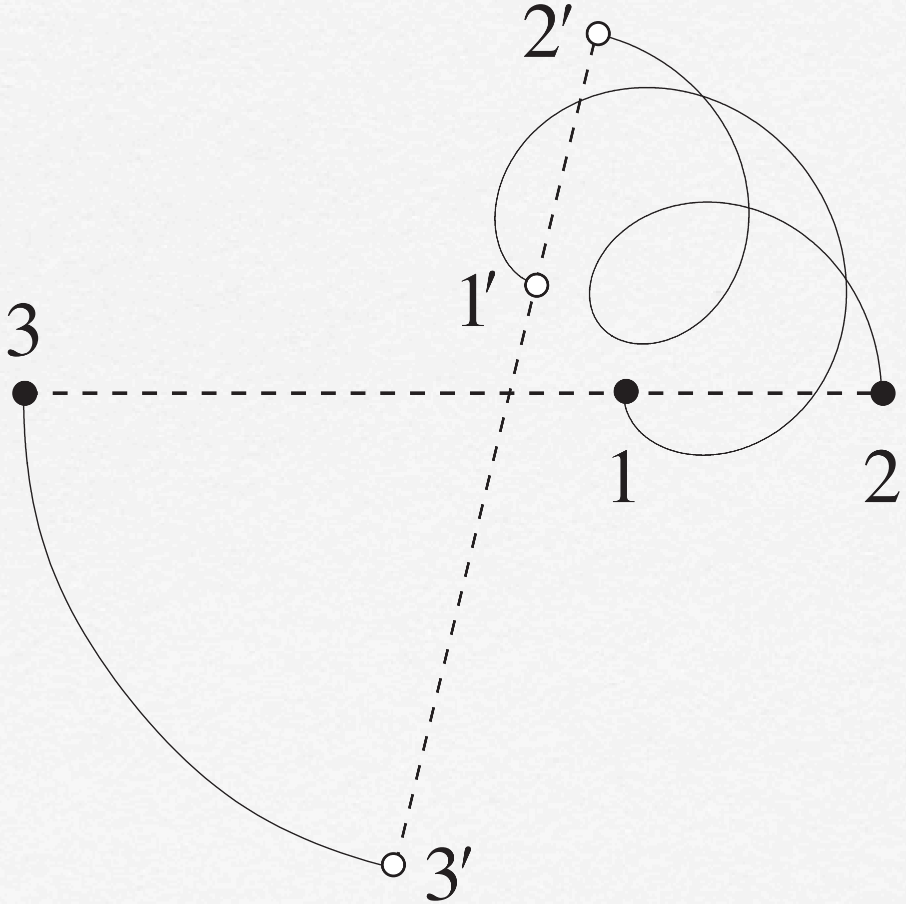
Phase plane 1:1:1

Notes:

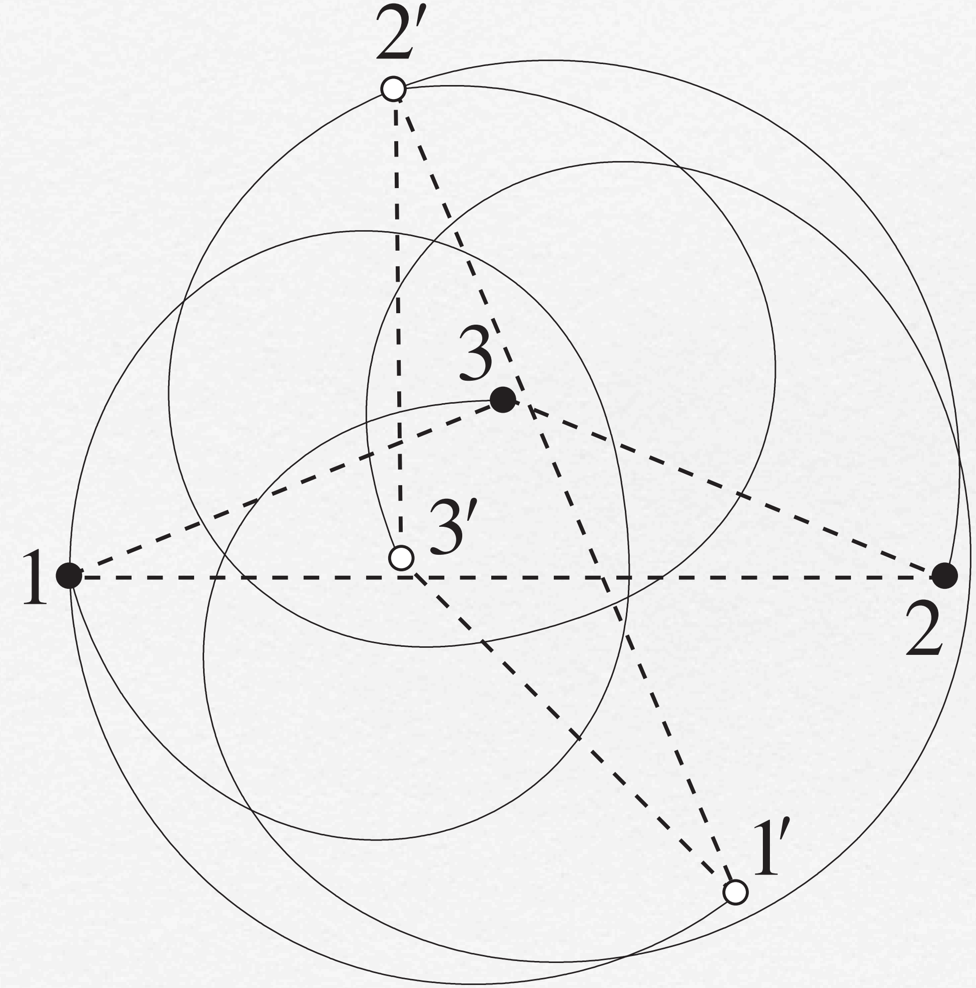
- b_1, b_2, b_3 proportional to $\frac{d_{23}^2}{\Gamma_1}$, $\frac{d_{31}^2}{\Gamma_2}$, $\frac{d_{12}^2}{\Gamma_3}$, respectively
- Only states inside bold circle are allowed
- Center corresponds to the equilateral triangle
- Boundary corresponds to collinear states

Motion of three identical vortices

(2,1)-like...



“collective”



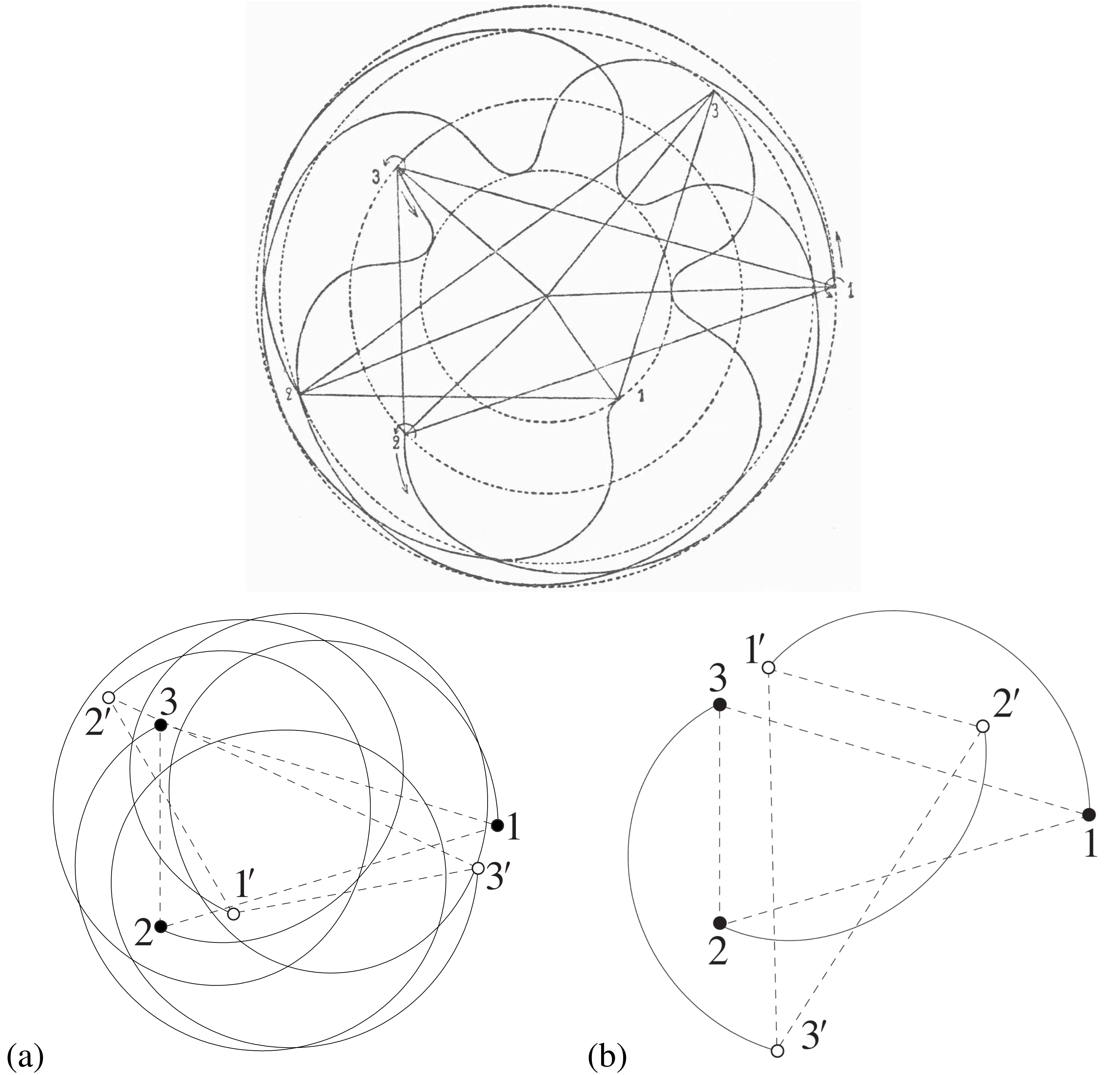
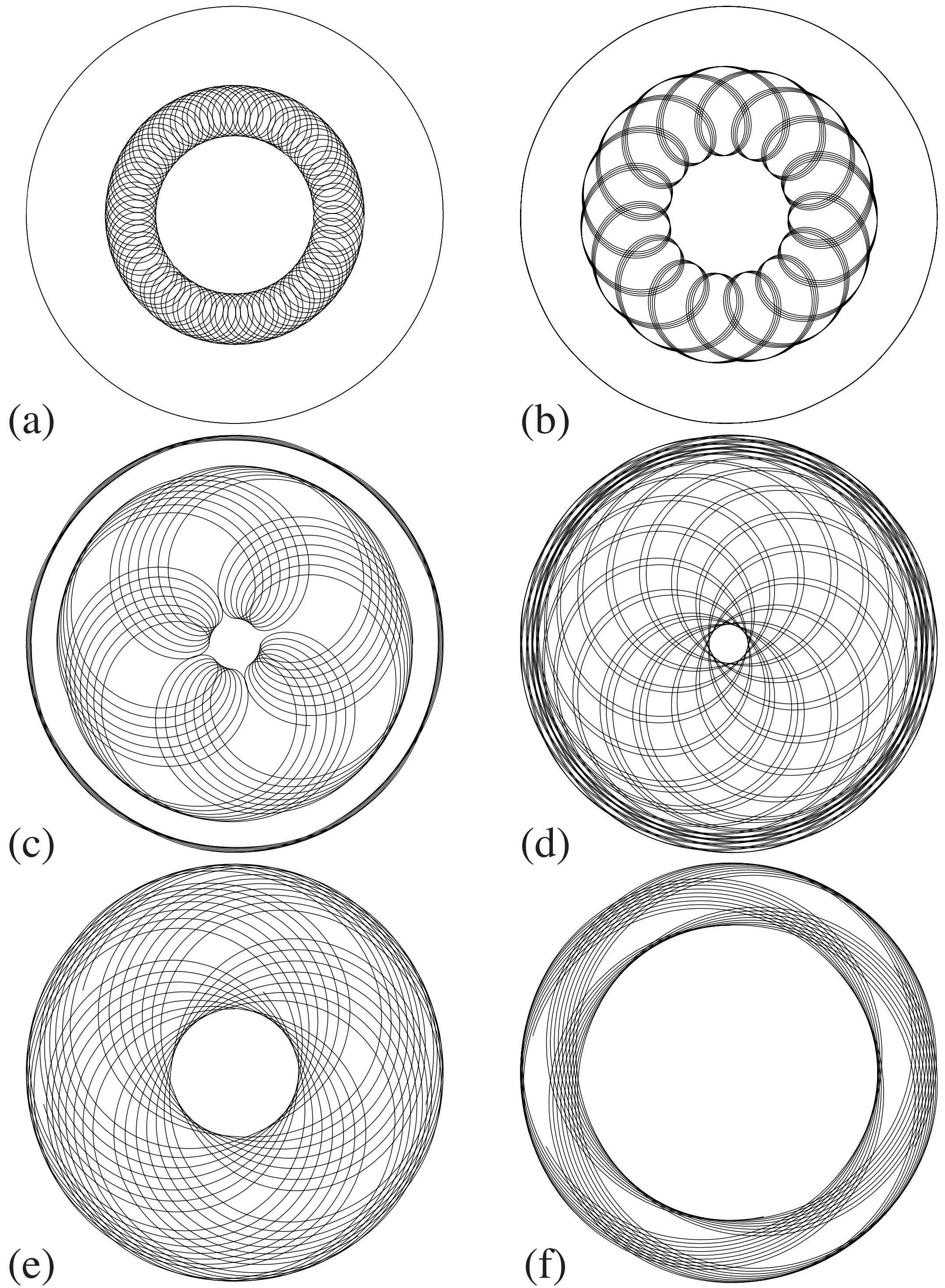
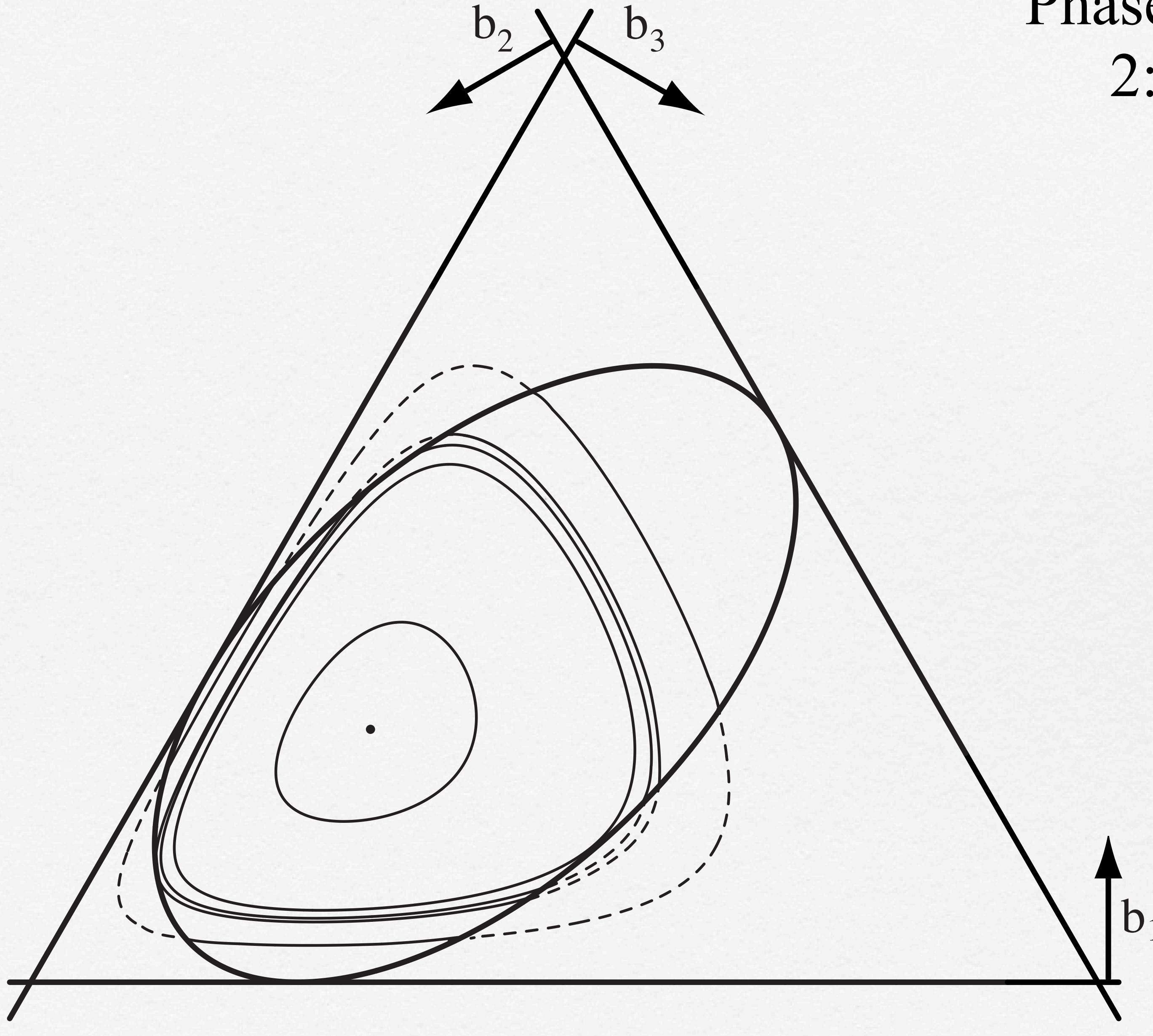
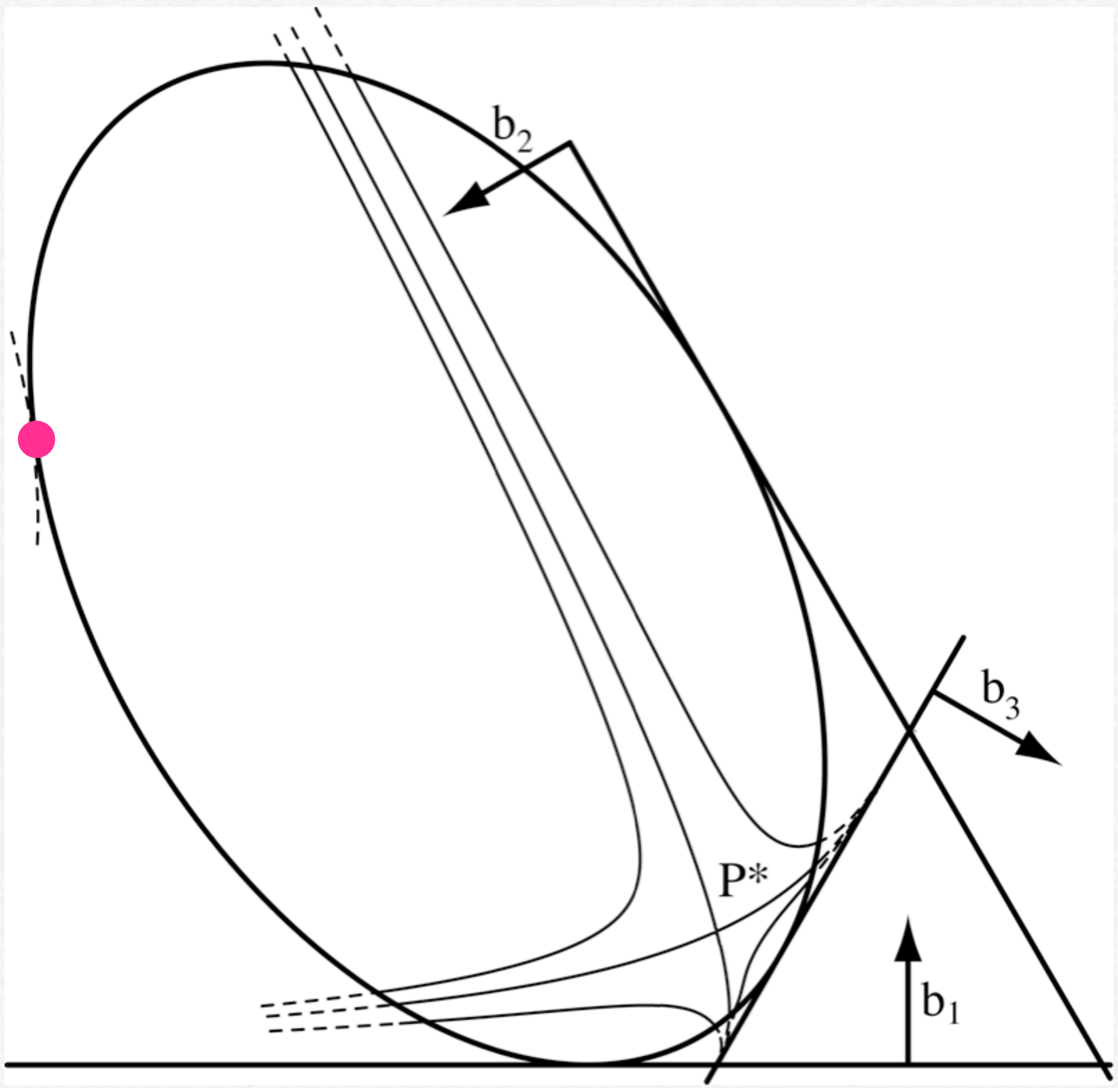


Figure 2: Vortex trajectories for three identical vortices, a case illustrated in detail by Gröbli in his thesis. (Top) Figure 4 from *Excerpt #17*; (a) a modern re-calculation; (b) one period of the area pulsation. Initial vortex positions (solid circles) are labeled 1, 2, 3, final positions (open circles) $1'$, $2'$, $3'$. The value of of λ^2 is $243/343 \approx 0.708\ldots$


(a)
(b)
(c)
(d)
(e)
(f)

Phase plane
2:1:3



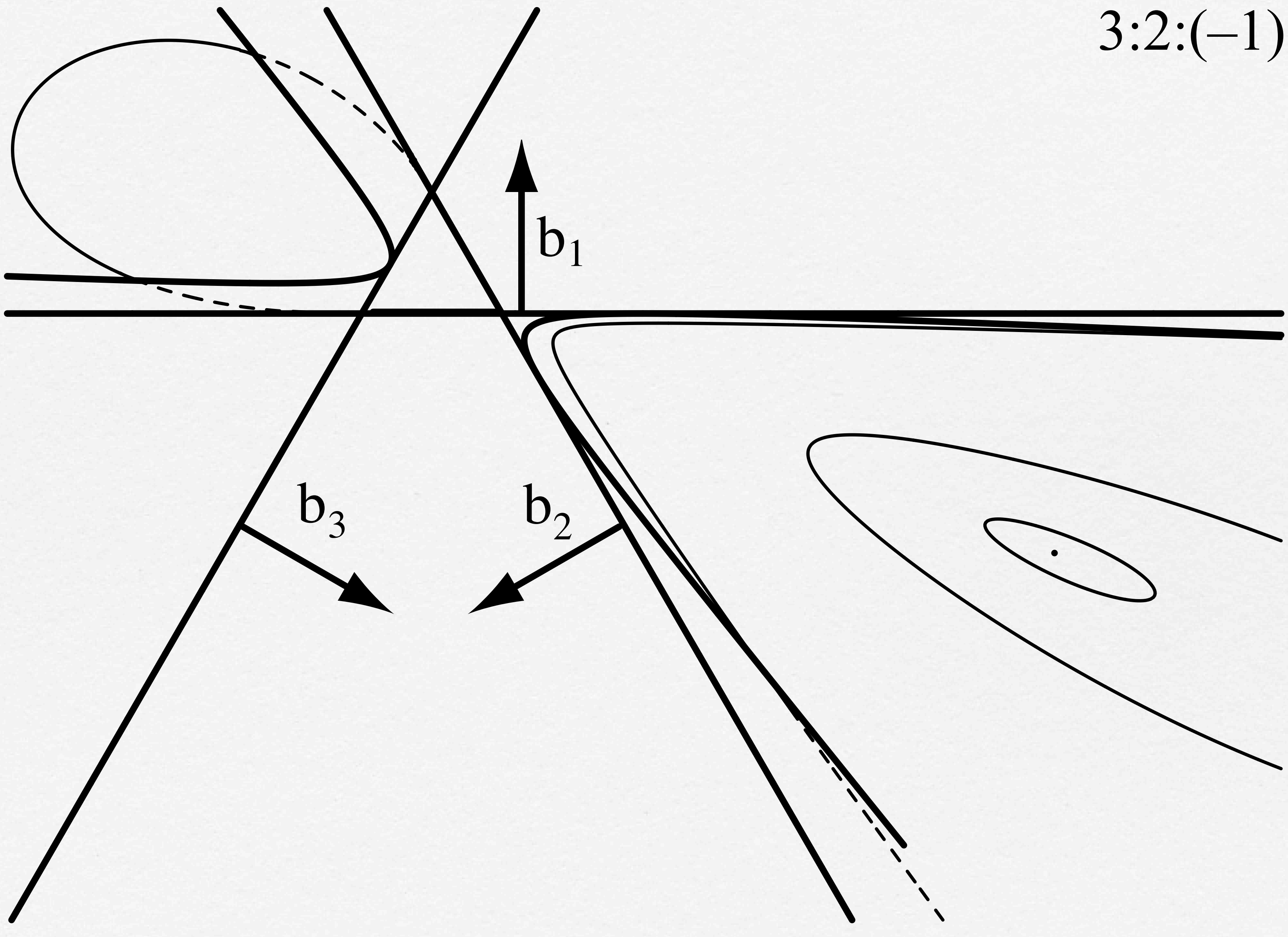


$\Gamma_1 : \Gamma_2 : \Gamma_3 = 2:1:(-4)$

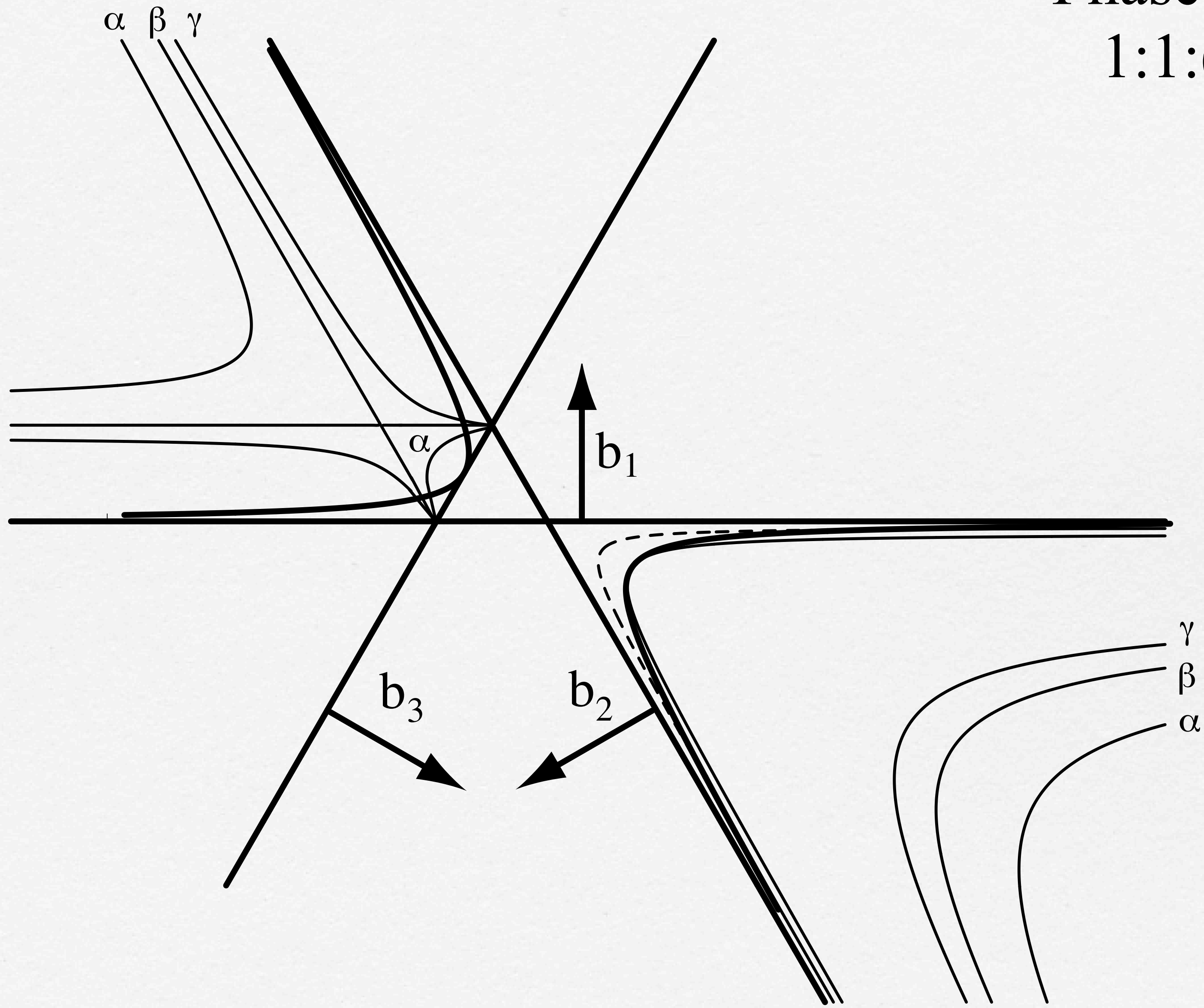
Notes:

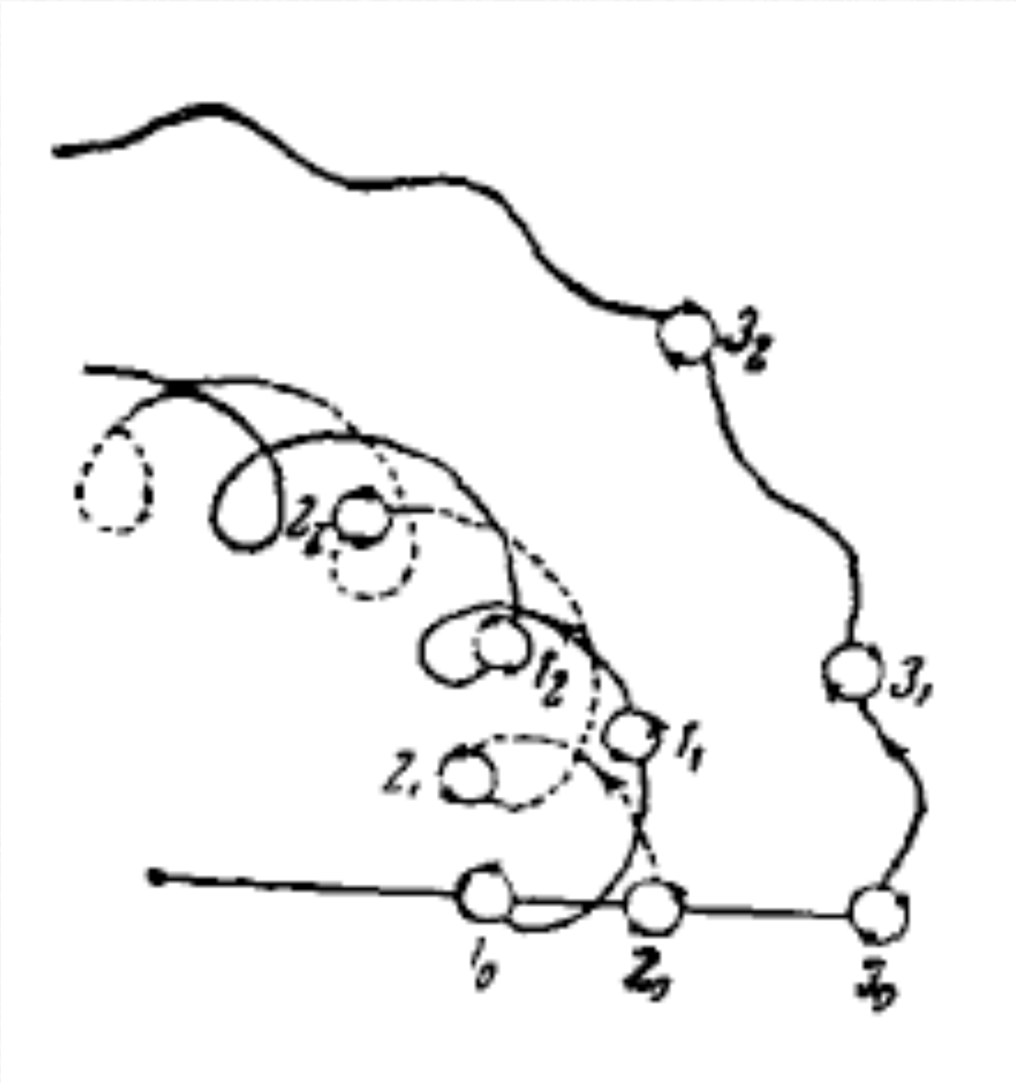
- b_1, b_2, b_3 proportional to $\frac{d_{23}^2}{\Gamma_1}$, $\frac{d_{31}^2}{\Gamma_2}$, $\frac{d_{12}^2}{\Gamma_3}$ respectively
- Only states inside ellipse are allowed
- P^* corresponds to the equilateral triangle
- Boundary corresponds to collinear states

Phase plane
3:2:(–1)



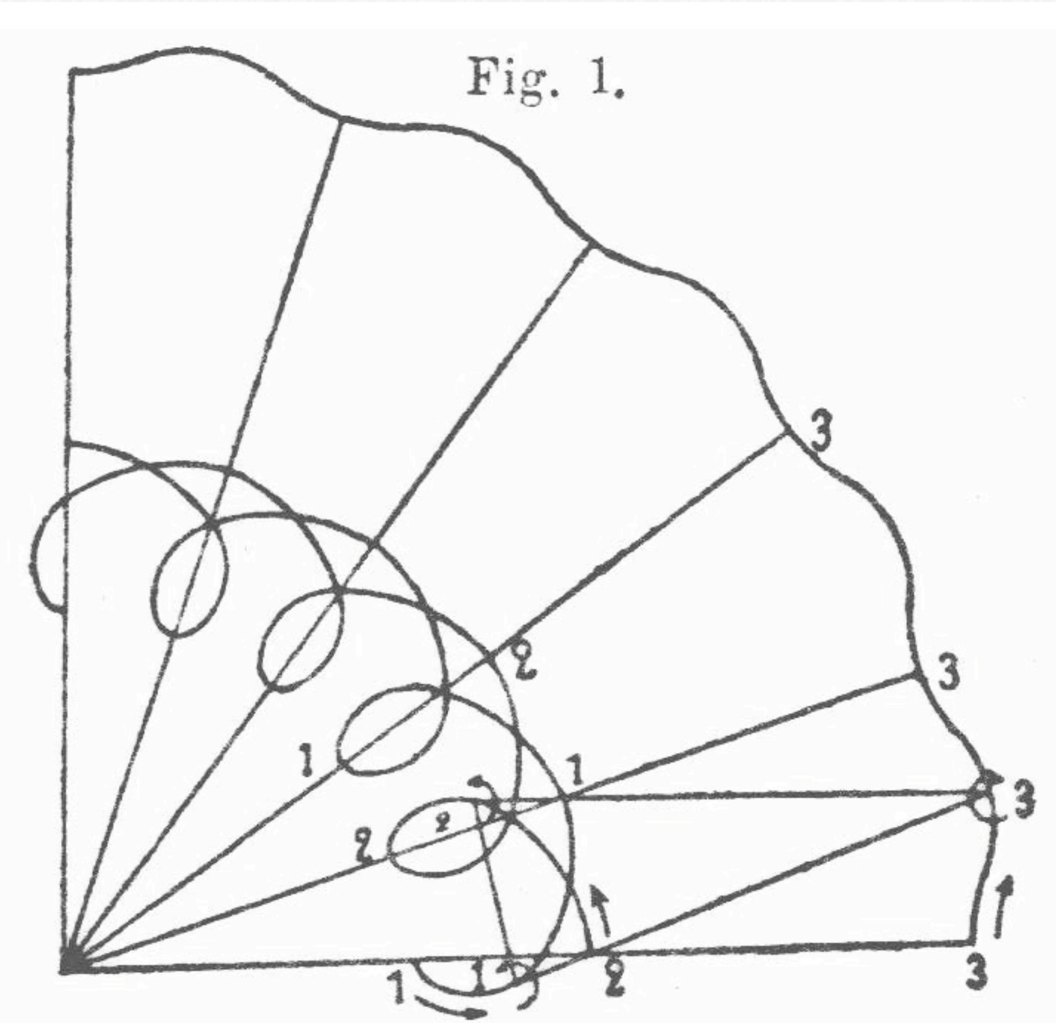
Phase plane
1:1:(–1)



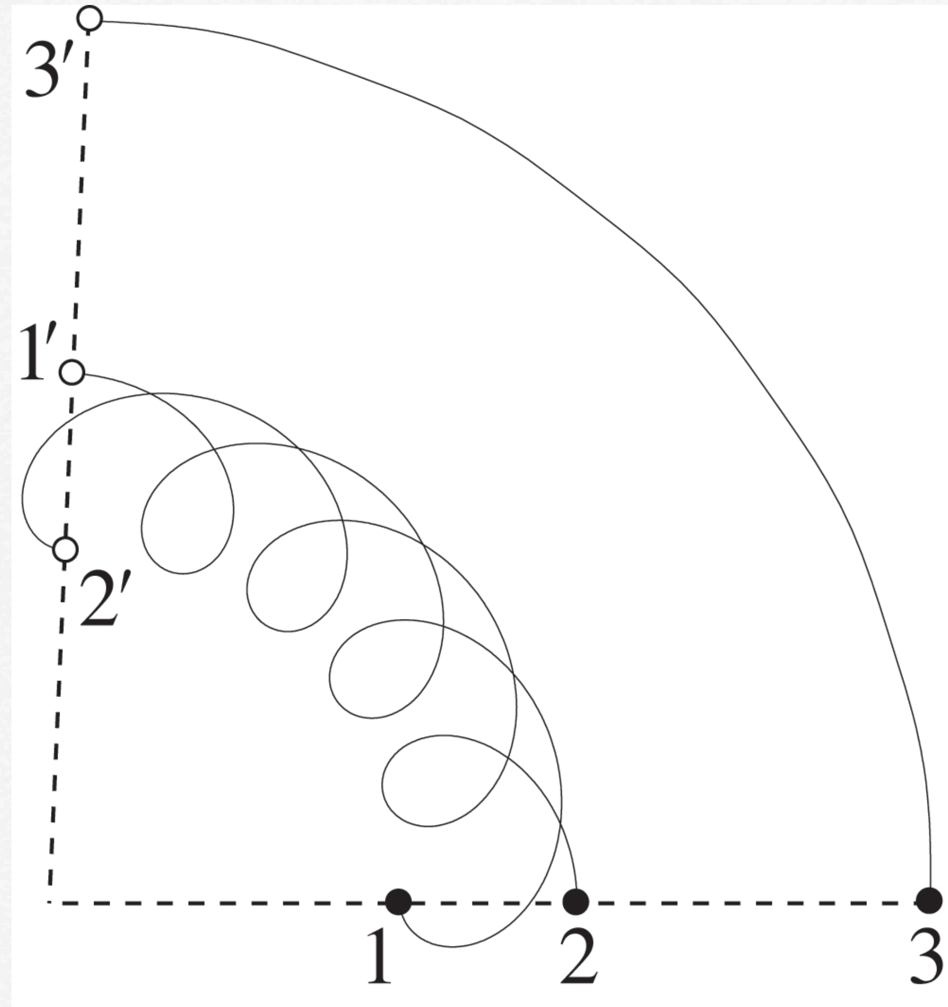


Nikolai Yegorovich Zhukovskii

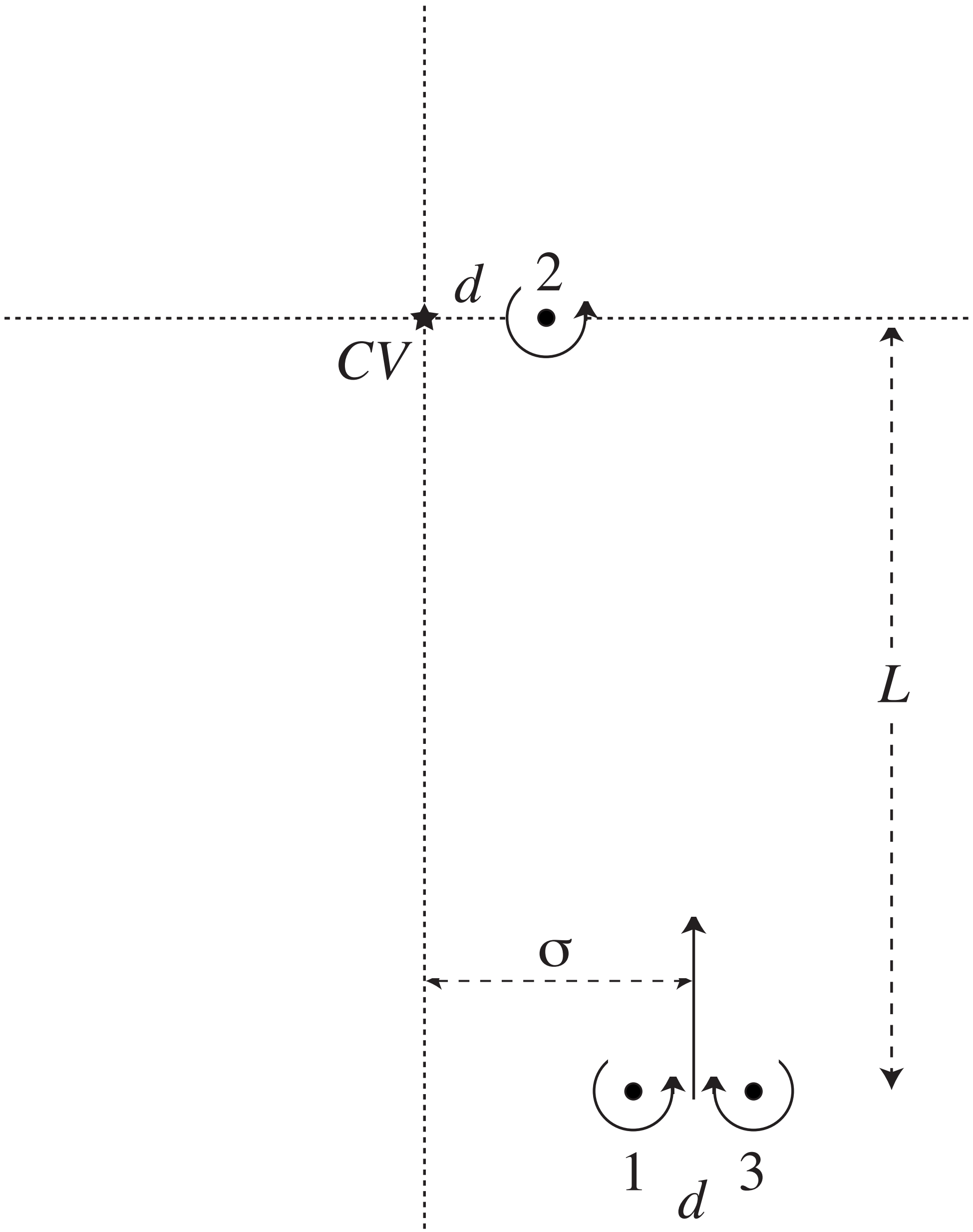
1847 - 1921



Gröbli 1877



Modern computation



Critical values

$$\sigma = 1, \tfrac{1}{2}, -\tfrac{7}{2}$$

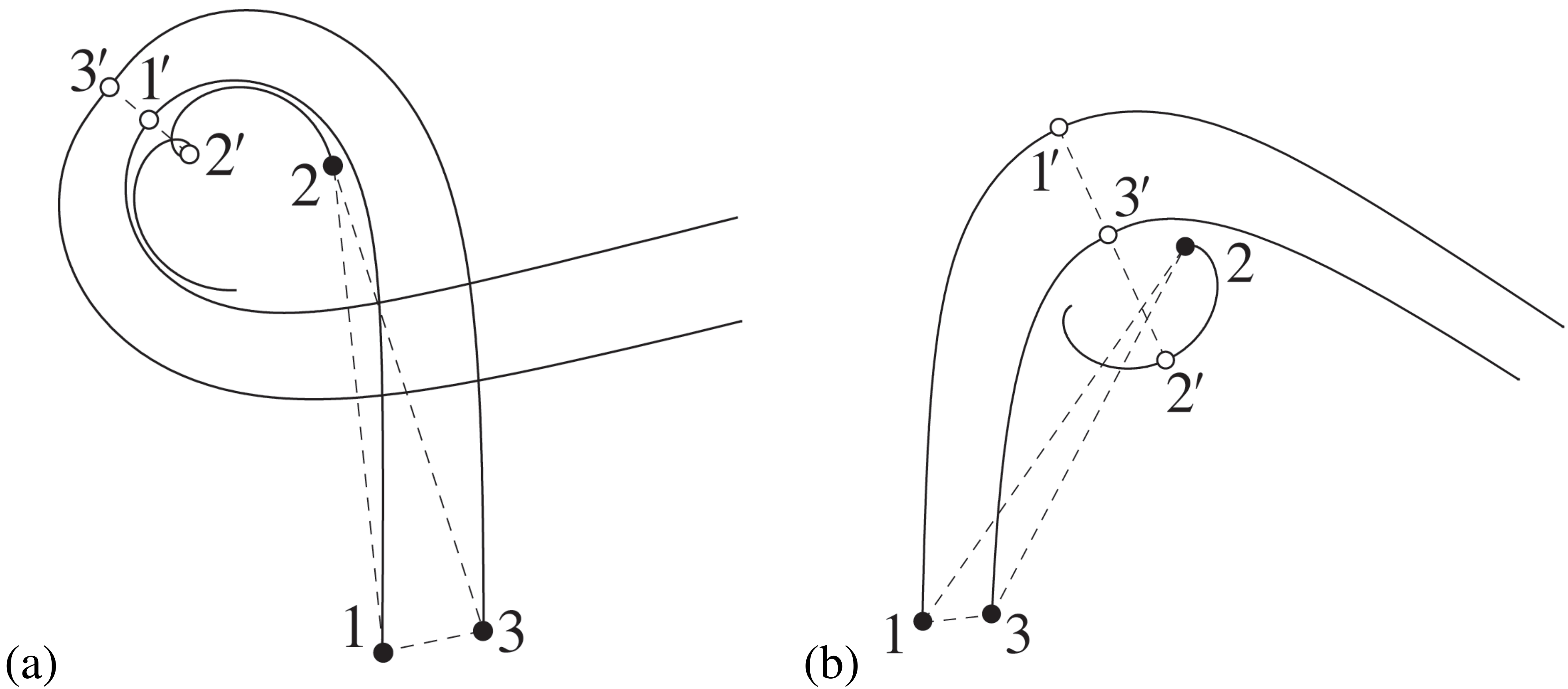


Figure 7: Examples of direct scattering close to trapping. (a) $\Lambda = -0.255$, (b) 2.01. Note that the orientation of the vortex triangle changes following collinearity. Substantial scattering angles are possible close to trapping.

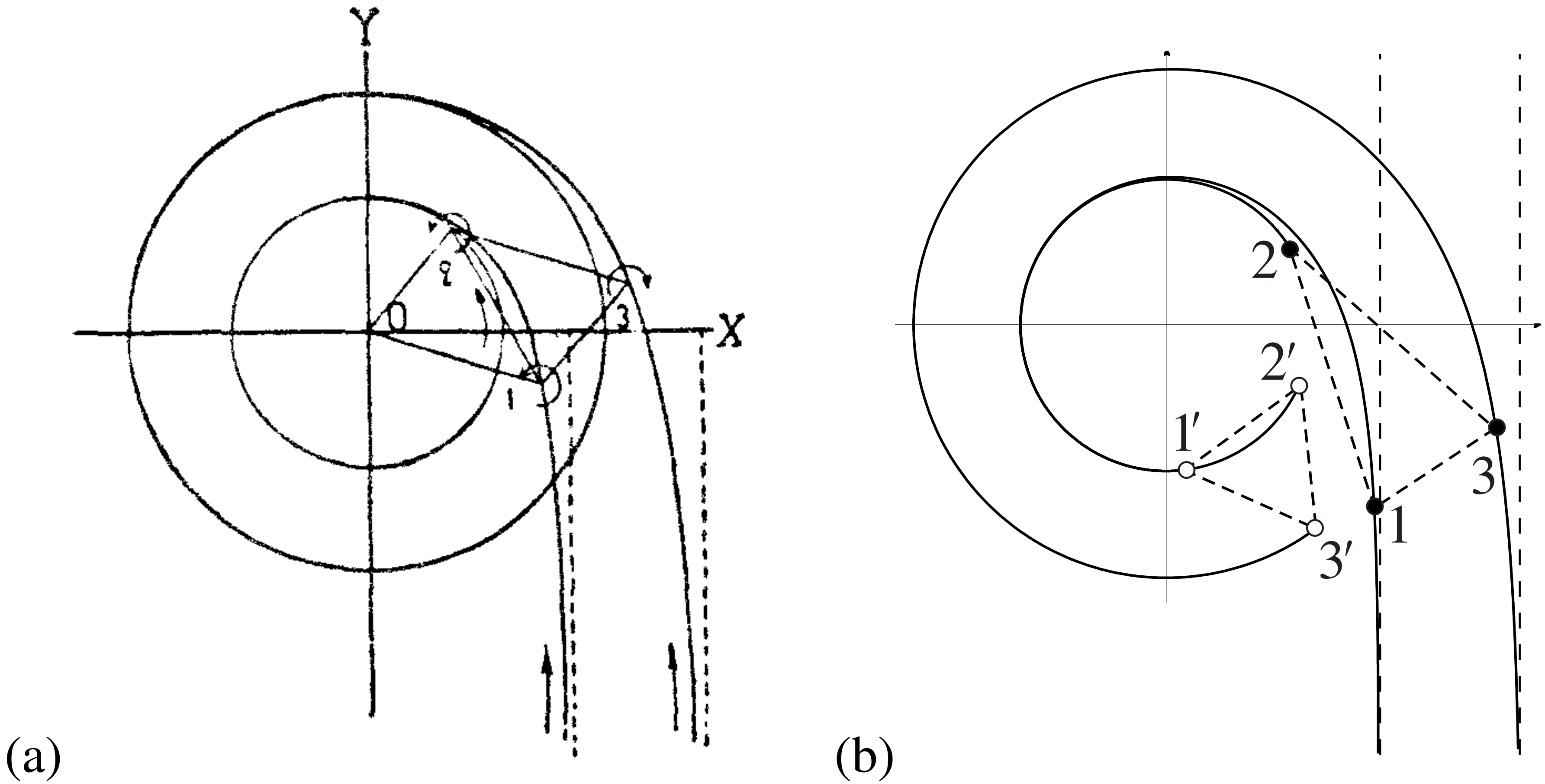


Figure 4: (a) Gröbli's illustration of trapping in an equilateral triangle configuration; (b) later parts of a recalculation using the initial conditions in Fig.2 with $\sigma = 1$, $L = 100$. (Trajectories of vortices 1 and 3 coming in from 'infinity' are shown; vortex 2 is only tracked from configuration 123 to $1'2'3'$.) Note that the asymptotes to the trajectories of vortices 1 and 3 are at $x_1 = \frac{3}{2}$ and $x_3 = \frac{5}{2}$, respectively and not, as one might expect, $x_1 = \frac{1}{2}$ and $x_3 = \frac{3}{2}$.

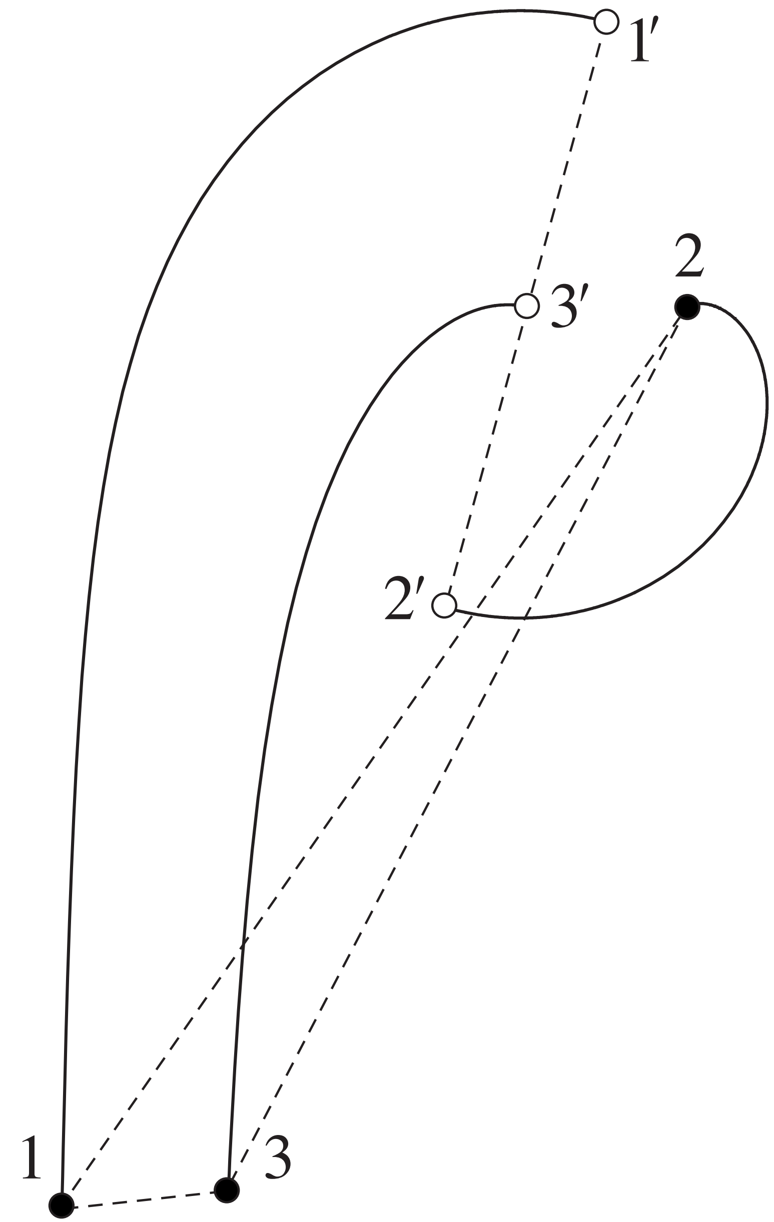


Figure 6: Trapping of the three-vortex system $(\Gamma, \Gamma, -\Gamma)$ in a collinear relative equilibrium. Motion is from configuration 123 to $1'2'3'$.

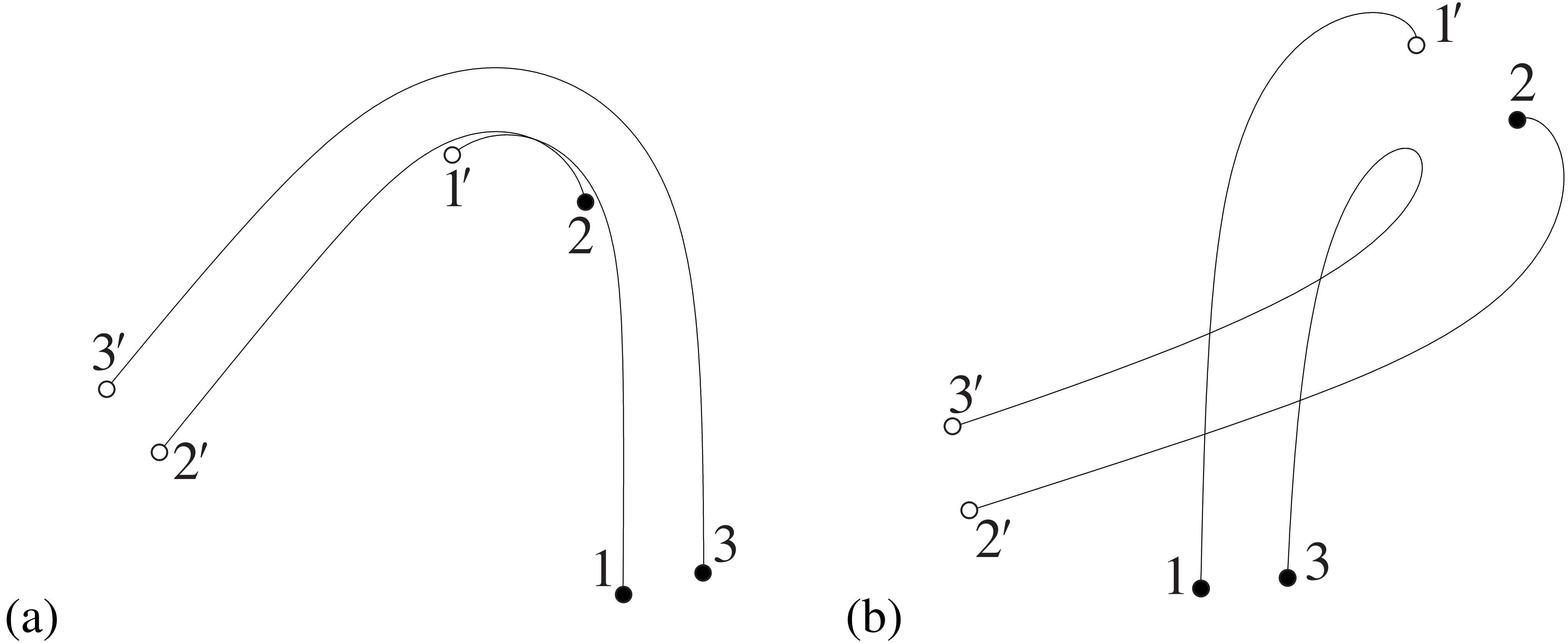
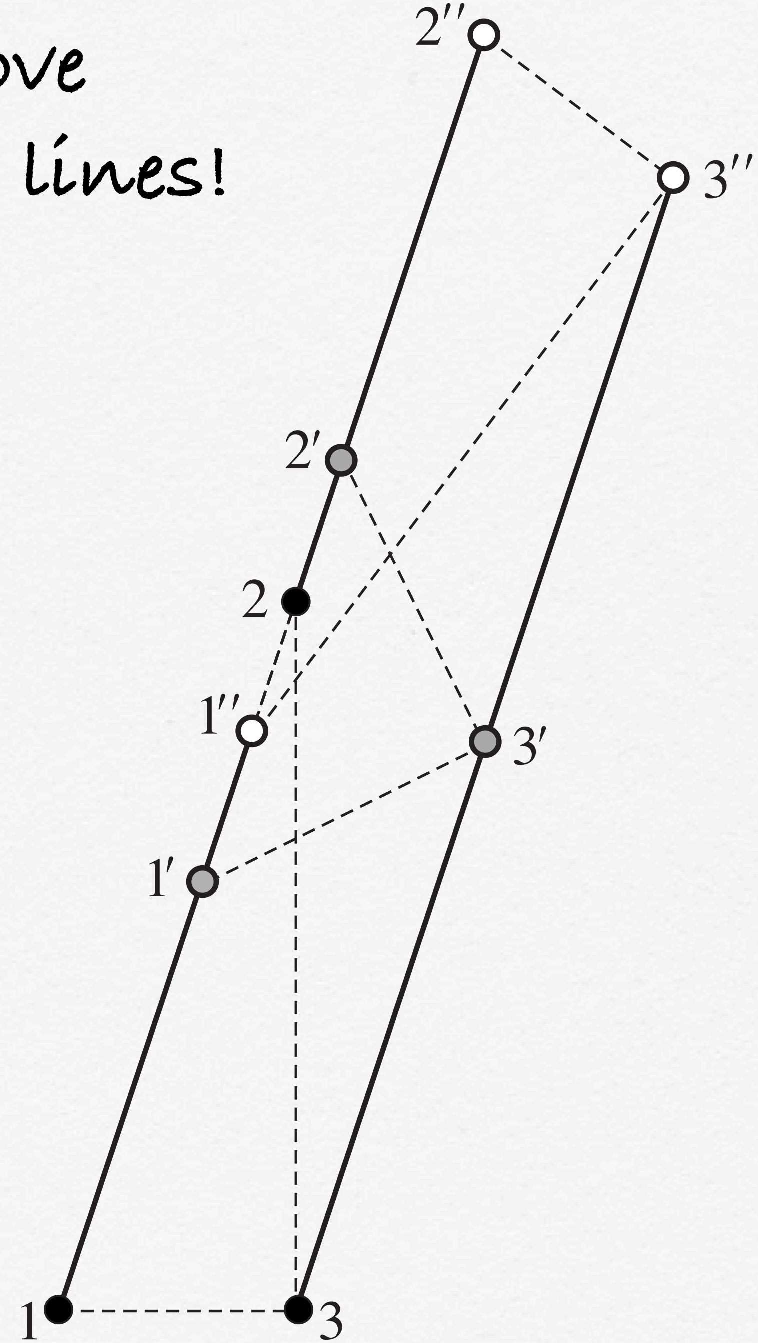
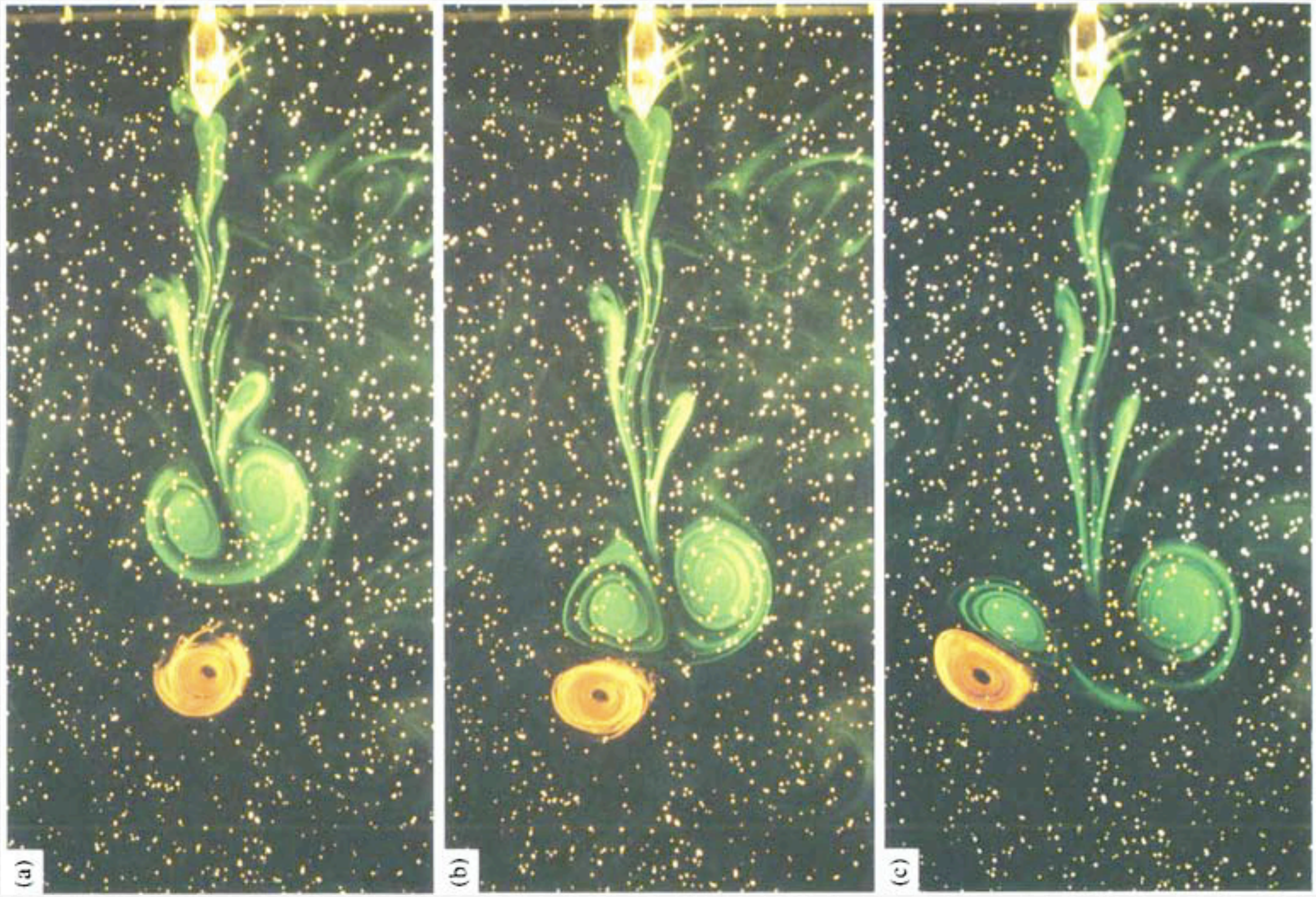


Figure 8: Examples of exchange scattering close to trapping. (a) $\Lambda = -0.245$, (b) 1.95. The vortices are never collinear. Substantial scattering angles are possible close to trapping.

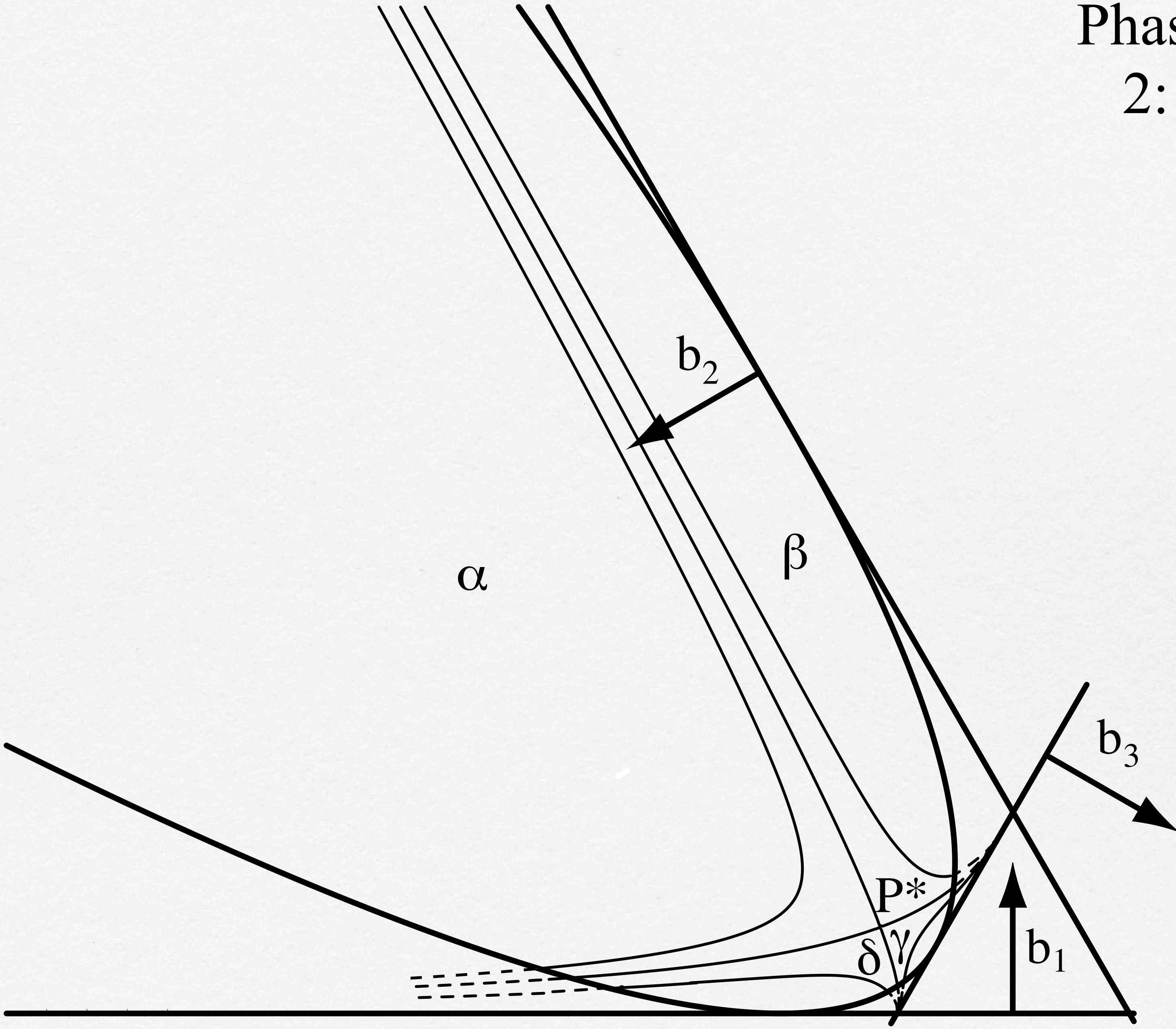
All vortices move along straight lines!





Interacting Two-dimensional Vortex Structures: Point Vortices, Contour Kinematics and Stirring Properties
V. V. MELESHKO & G. J. F. VAN HEIJST, Chaos, Solitons & Fractals, **4** (6), 977-1010 (1994)

Phase plane
2:1:(–3)



The case of vanishing total circulation

$$\Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3 = X + iY \qquad \Gamma_1 + \Gamma_2 + \Gamma_3 = 0$$

Set

$$z_1 - z_2 = Z$$

Then, all vortex separations can be stated in terms of Z

$$z_2 - z_3 = \frac{\Gamma_1 Z - \Xi}{\Gamma_3}$$

$$z_3 - z_1 = \frac{\Gamma_2 Z + \Xi}{\Gamma_3}$$

where $\Xi = X + iY$

Rott, Aref (1989)

Rott, Aref (1989)

$$\dot{z}_1^* = \frac{1}{2\pi i}\left(\frac{\Gamma_2}{z_1 - z_2} + \frac{\Gamma_3}{z_1 - z_3}\right)$$

$$\dot{z}_2^* = \frac{1}{2\pi i}\left(\frac{\Gamma_1}{z_2 - z_1} + \frac{\Gamma_3}{z_2 - z_3}\right)$$

$$\dot{z}_3^* = \frac{1}{2\pi i}\left(\frac{\Gamma_1}{z_3 - z_1} + \frac{\Gamma_2}{z_3 - z_2}\right)$$

Equations of motion combine to produce

$$\dot{Z}^* = -\frac{\Gamma_3^2}{2\pi i}\left(\frac{\Gamma_3^{-1}}{Z} + \frac{\Gamma_2^{-1}}{Z + \frac{\Xi}{\Gamma_2}} + \frac{\Gamma_1^{-1}}{Z - \frac{\Xi}{\Gamma_1}}\right)$$

$$\dot{Z}^* = -\frac{\Gamma_3^2}{2\pi i}\left(\frac{\Gamma_3^{-1}}{Z} + \frac{\Gamma_2^{-1}}{Z + \dfrac{\Xi}{\Gamma_2}} + \frac{\Gamma_1^{-1}}{Z - \dfrac{\Xi}{\Gamma_1}}\right)$$

Interpret as advection of particle at Z by three stationary vortices:

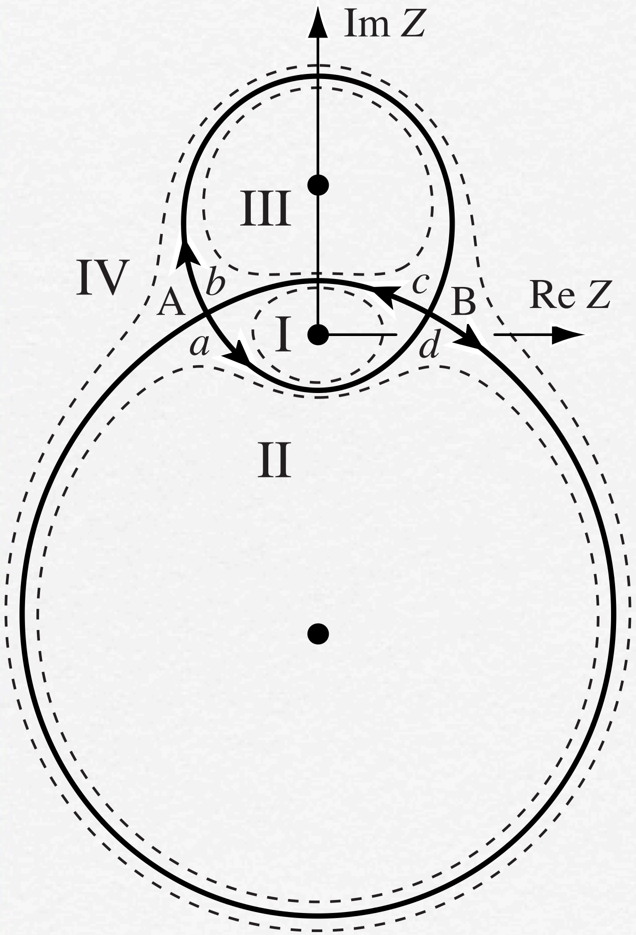
Circulation	Position
Γ_3^{-1}	Origin
Γ_2^{-1}	$-\Xi/\Gamma_2$
Γ_1^{-1}	Ξ/Γ_1

Rotate coordinates so X = 0

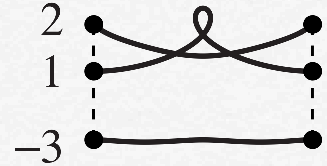
Note:
$\Gamma_1^{-1} + \Gamma_2^{-1} + \Gamma_3^{-1} \neq 0$

Rott, Aref (1989)

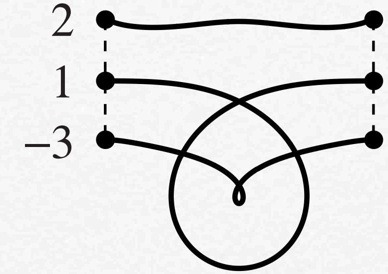
$\Gamma_1 : \Gamma_2 : \Gamma_3 = 2:1:(-3)$

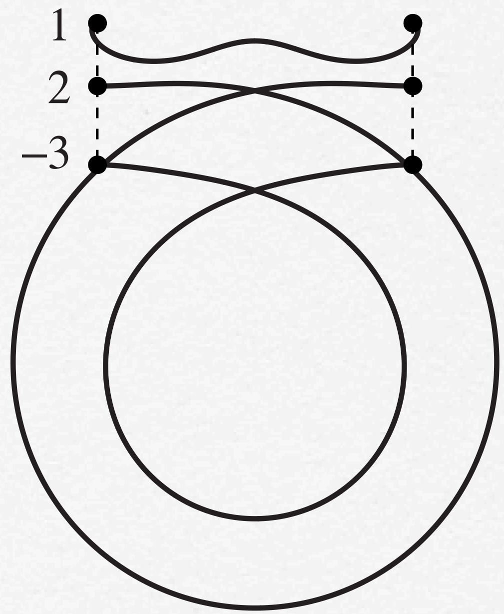


I

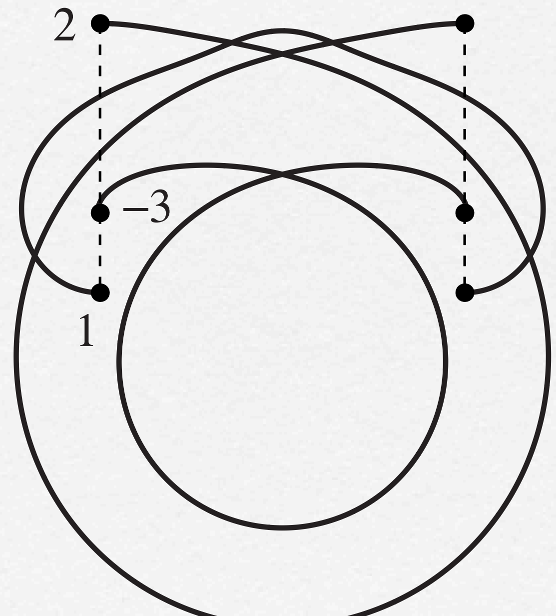


III





II

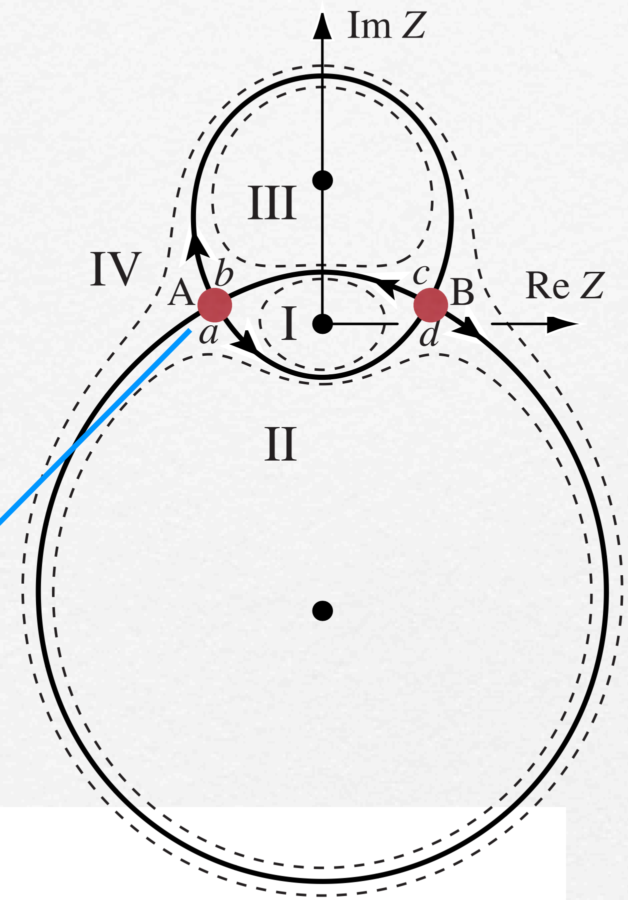


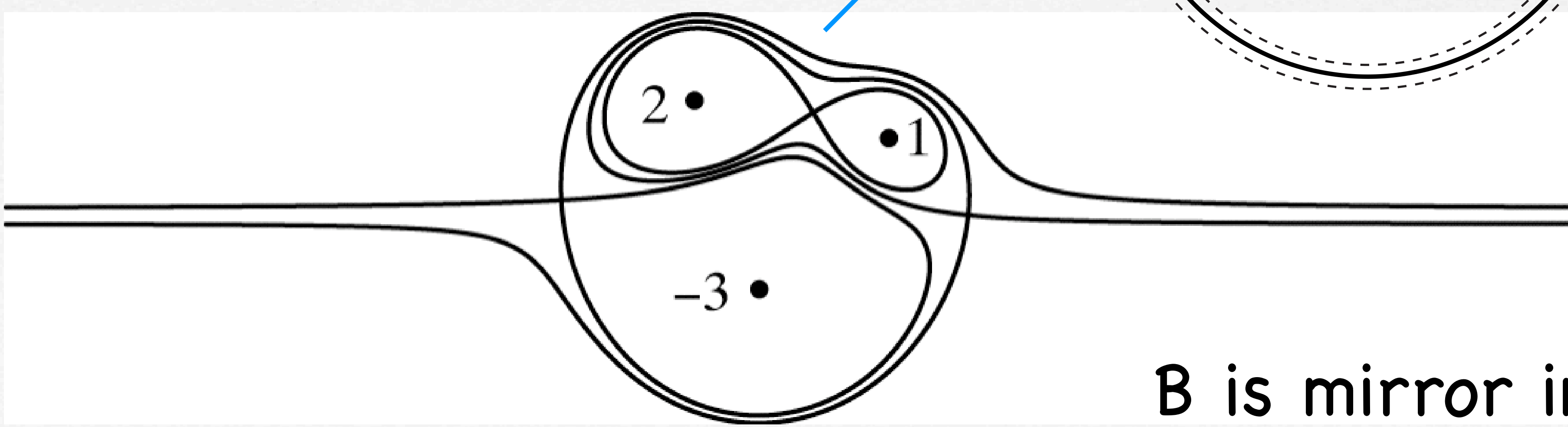
IV

Rott, Aref (1989)

Trajectory plots by M. A. Stremler

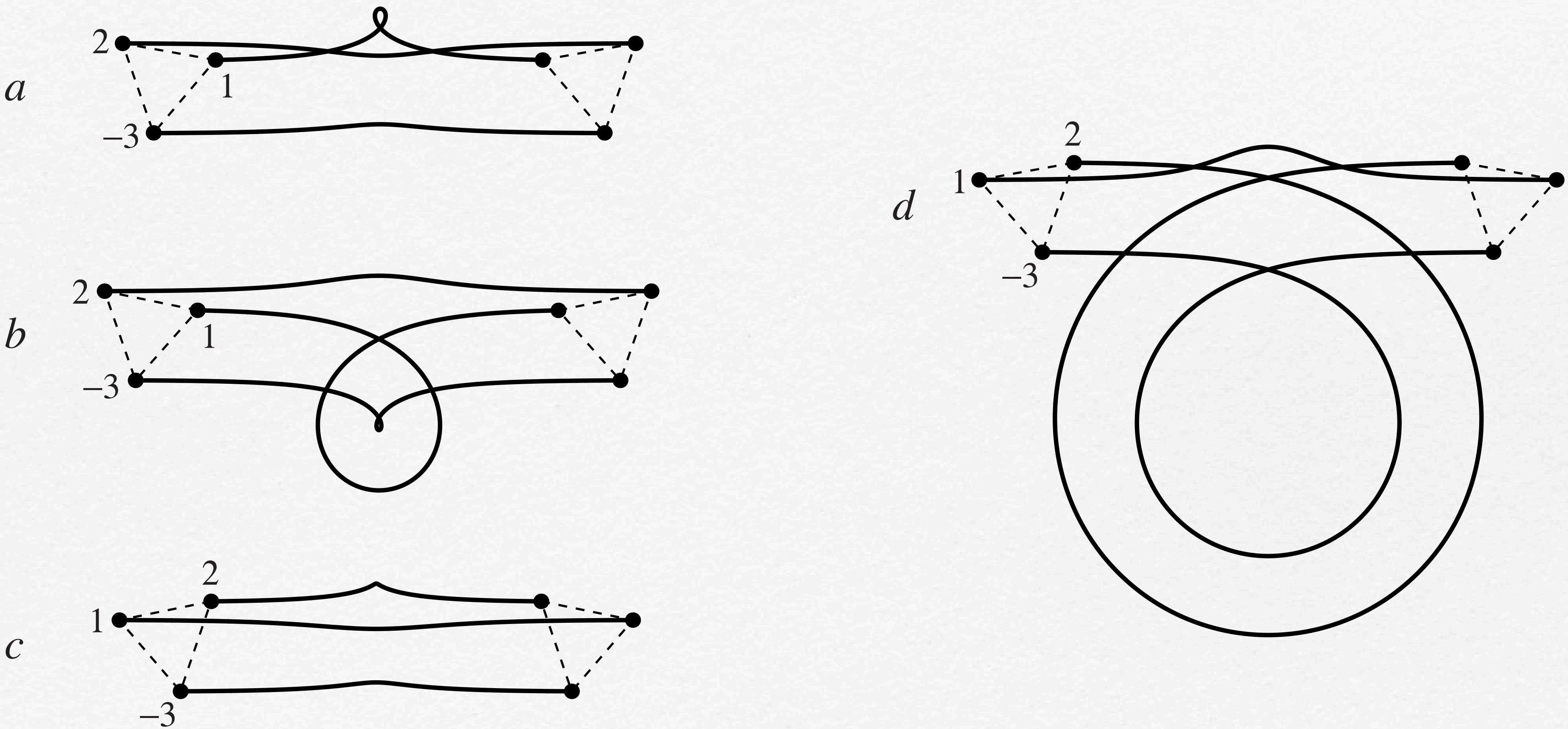
Saddle points correspond to translating relative equilibria





B is mirror image

Saddle connections flip vortex triangle orientation



$\Gamma_1 : \Gamma_2 : \Gamma_3 = 2 : 1 : (-3)$

Trajectory plots by M. A. Stremler

The zero-impulse case: $\Xi = 0$

- Best known example: Vortex Tripole
- Lab version: Kloosterziel & van Heijst, J. Fluid Mech. 223 (1991) 1
- Point vortex counterpart: Three vortices on a line with strengths in the ratio -1:2:(-1)
- Configuration rotates about central vortex

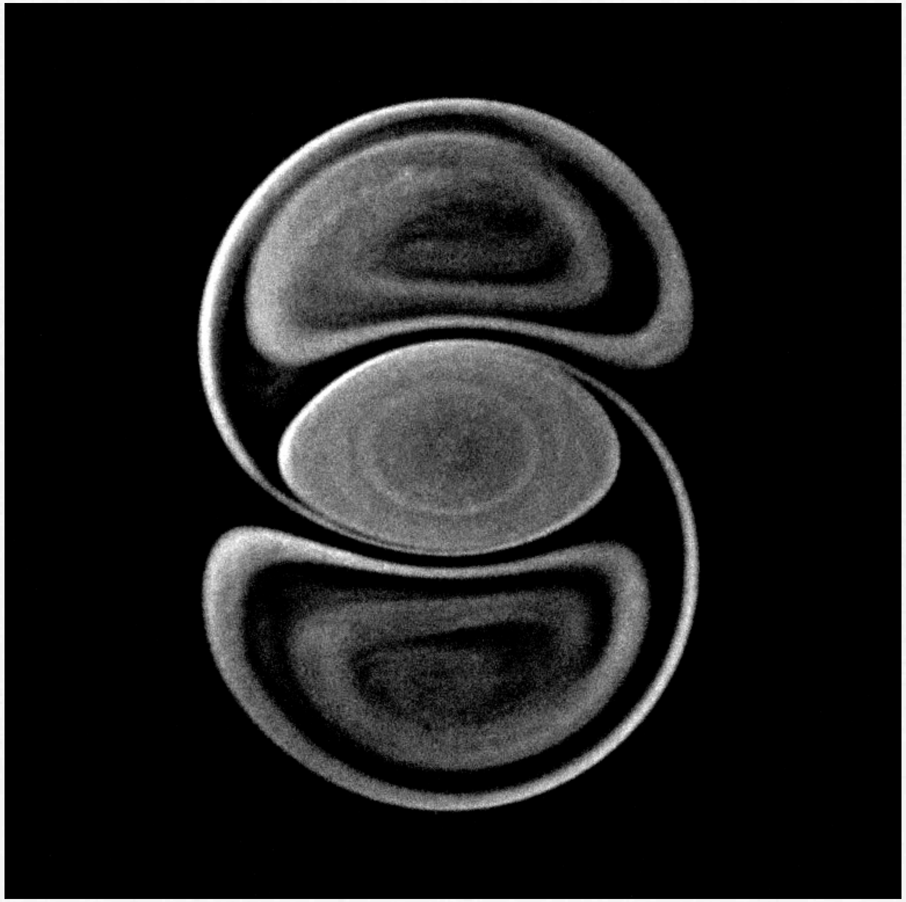


Photo courtesy of G. J. van Heijst

The "restricted" four-vortex problem

- Advection of a particle by flow field due to three interacting vortices
- Dynamical system:

$$\dot{z}^*_\alpha = \frac{1}{2\pi i} \sum_{\beta=1}^{3}{}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}$$

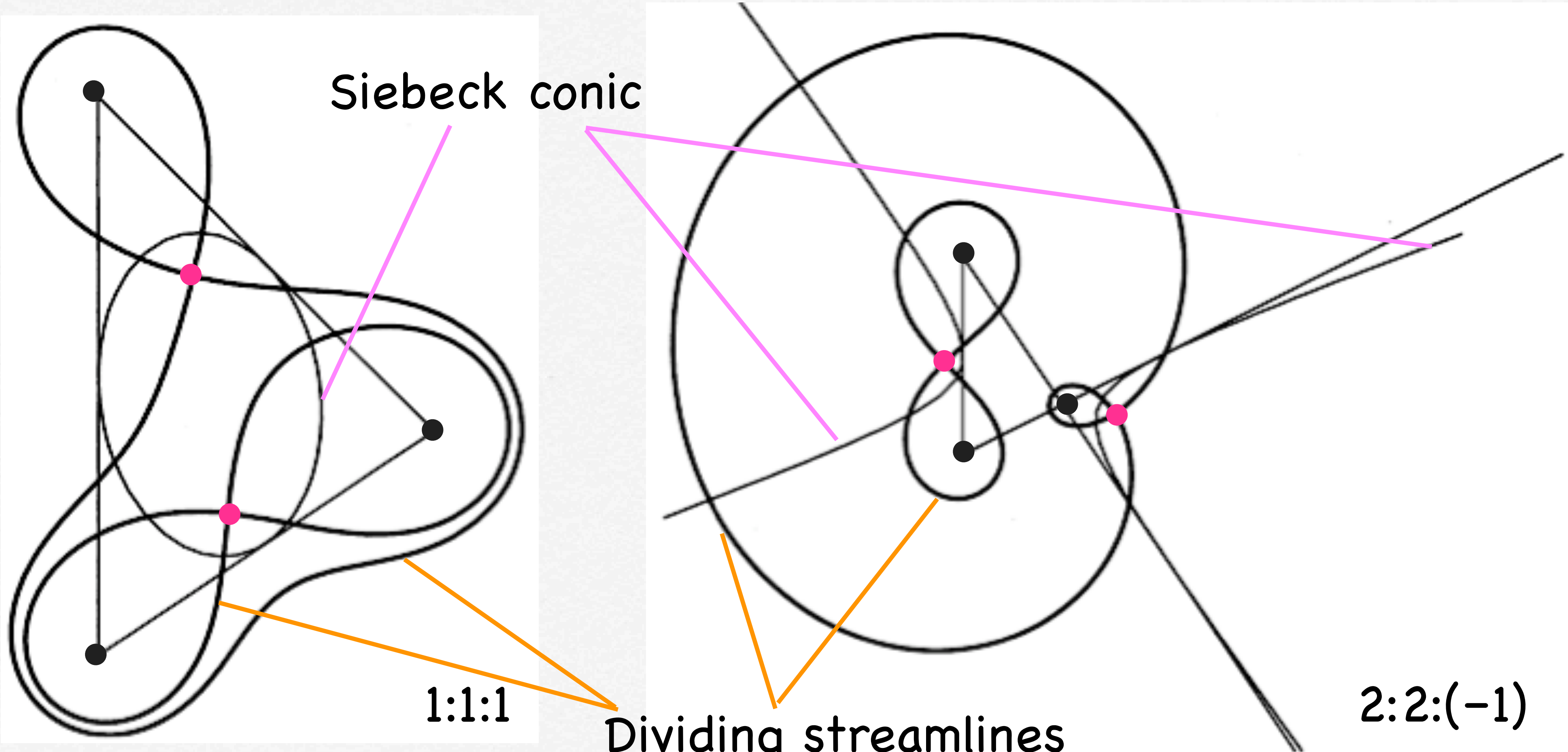
$$\dot{z}^* = \frac{1}{2\pi i} \sum_{\alpha=1}^{3} \frac{\Gamma_\alpha}{z - z_\alpha}$$

z-motion is non-integrable in general; simple example of chaotic advection

H. Aref, "The development of chaotic advection." Physics of Fluids 14, 1315-1325 (2002).

Stagnation points as foci

- Vortices
- Stagnation points



H. Aref & M. Brøns, "On stagnation points and streamline topology in vortex flows."
J. Fluid Mech. 370, 1–27 (1998)

Integrable four-vortex motion

In general, four-vortex motion is non-integrable.
However, recall earlier results:

$$[X, Y] = \sum_{\alpha=1}^{N} \Gamma_\alpha, \; [X, I] = 2Y, \quad [Y, I] = -2X$$

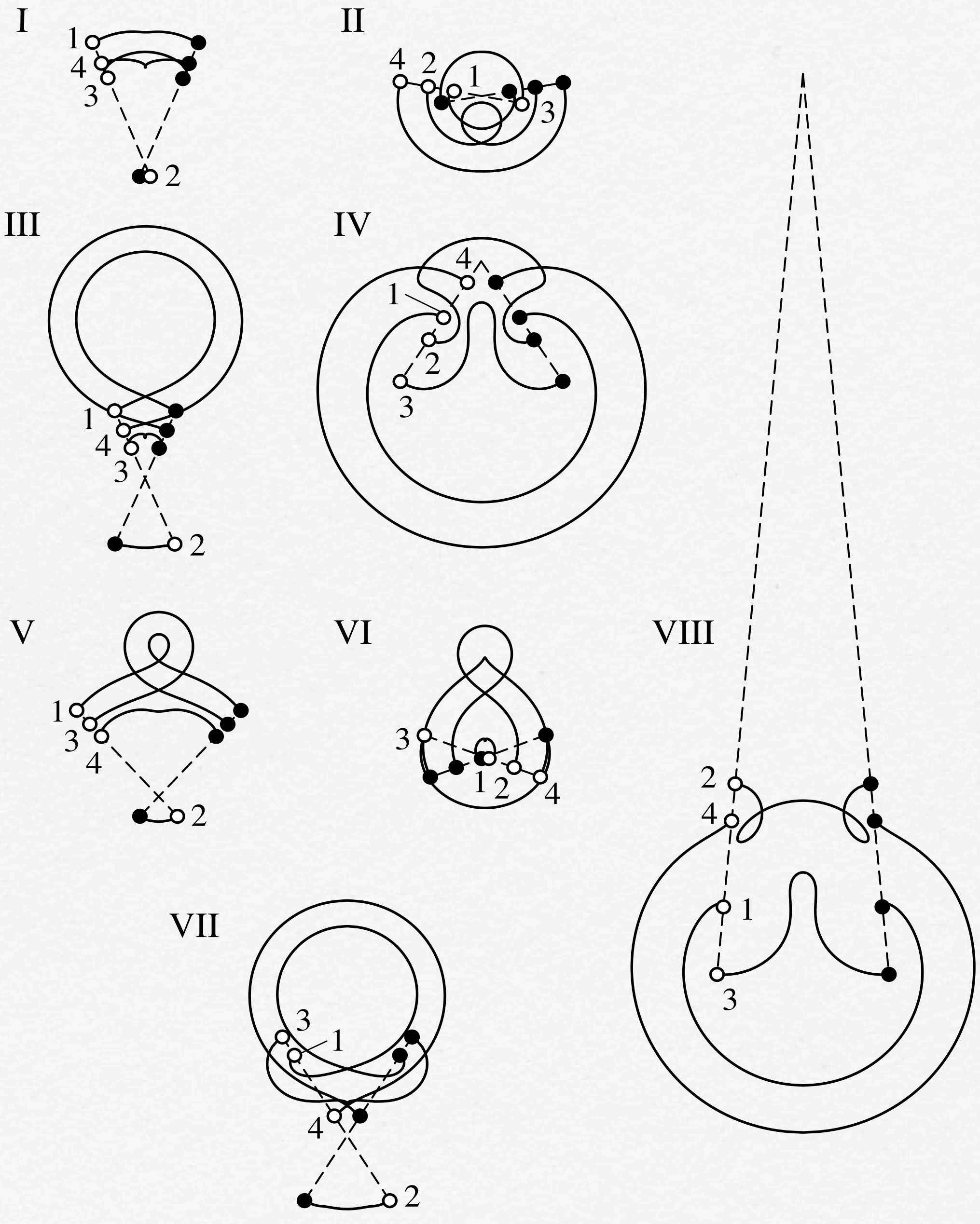
Thus, if X = Y = 0, and $\sum_{\alpha=1}^{N} \Gamma_\alpha = 0$, X, Y, I and H

are 4 independent integrals in involution...

...and the four-vortex problem is integrable!

Eckhardt (1988); Aref & Stremler (1999)

Sample trajectories for the integrable four-vortex system with circulations 4:1:(−2):(−3)



From H. Aref & M. A. Stremler, "Four-vortex motion with zero total circulation and impulse." Physics of Fluids 11, 3704–3715 (1999)

Three vortices in a periodic strip or box

- Think of as vortex rows/lattices (strip/box)
- System is still Hamiltonian
- Translational invariance is maintained
- Rotational invariance is lost
- Thus, H, X and Y are general integrals

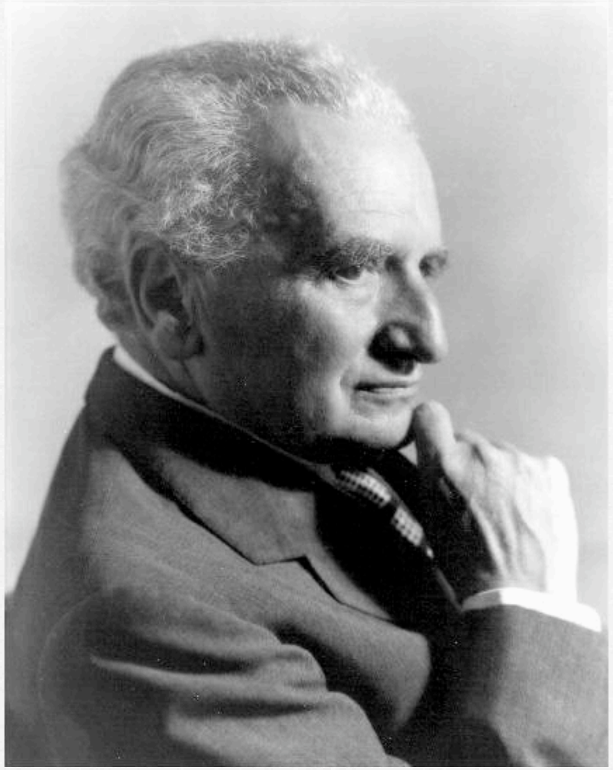
$$[H, X] = [H, Y] = 0, \; [X, Y] = \sum_{\alpha=1}^{N} \Gamma_\alpha$$

- Conclusion: Three-vortex system with zero net circulation in a periodic strip/box* is integrable

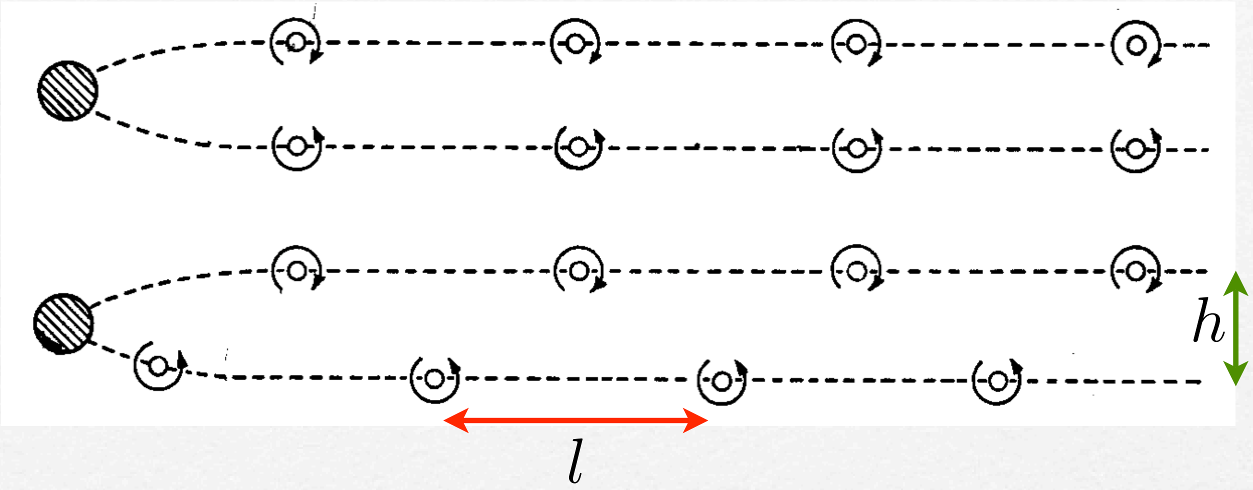
*) Double periodicity of flow field assures zero net circulation

Kármán vortex street

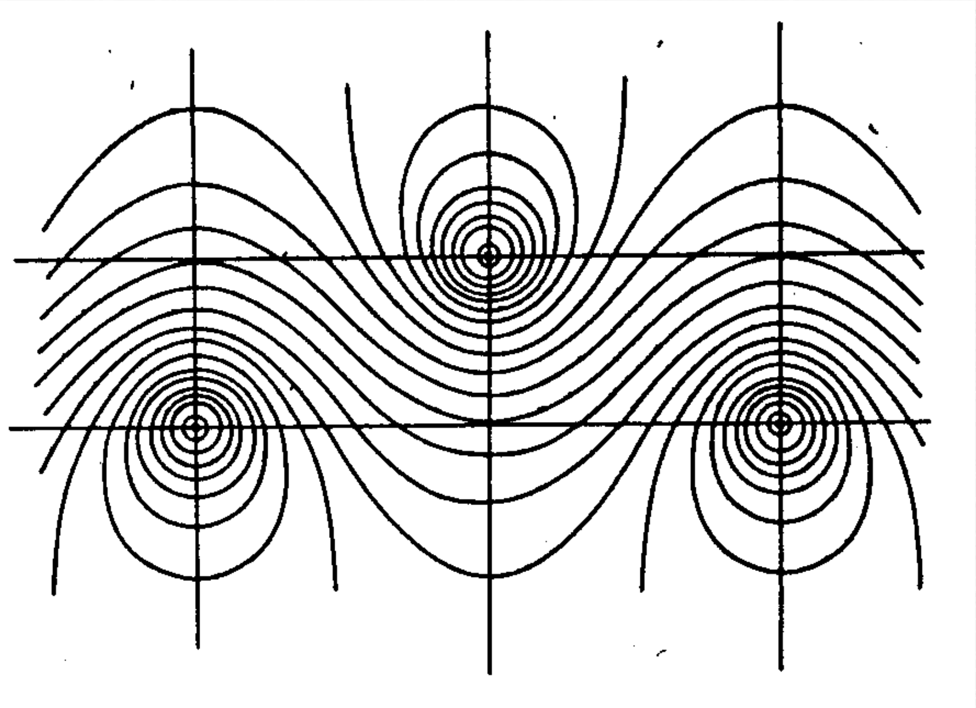
- Steady state: two-vortices-in-a-strip problem
- Stability: four-vortices-in-a-strip (Domm 1959)



Theodore von Kármán
(1881-1963)



$$\cosh \frac{\pi h}{l} = \sqrt{2}$$



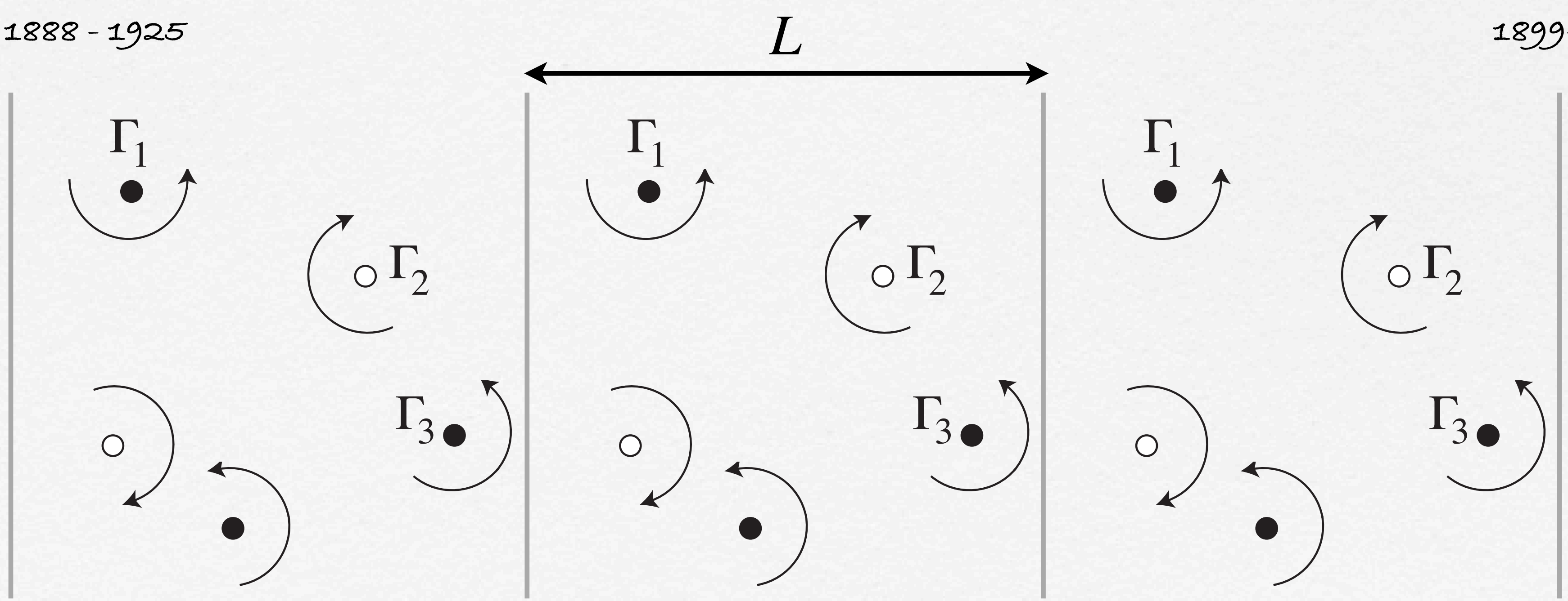
$$\overline{\frac{\mathrm{d}z_\alpha}{\mathrm{d}t}} = \frac{1}{2Li}\sum_{\beta=1}^{N}{}'\,\Gamma_\beta \cot\left[\frac{\pi(z_\alpha - z_\beta)}{L}\right]$$

Friedmann, A. & Polubarinova, P. 1928 Über fortschreitende Singularitäten der ebenen Bewegung einer inkompressiblen Flüssigkeit. *Recueil de Géophysique*, Tome V, Fascicule II, Leningrad, pp. 9–23

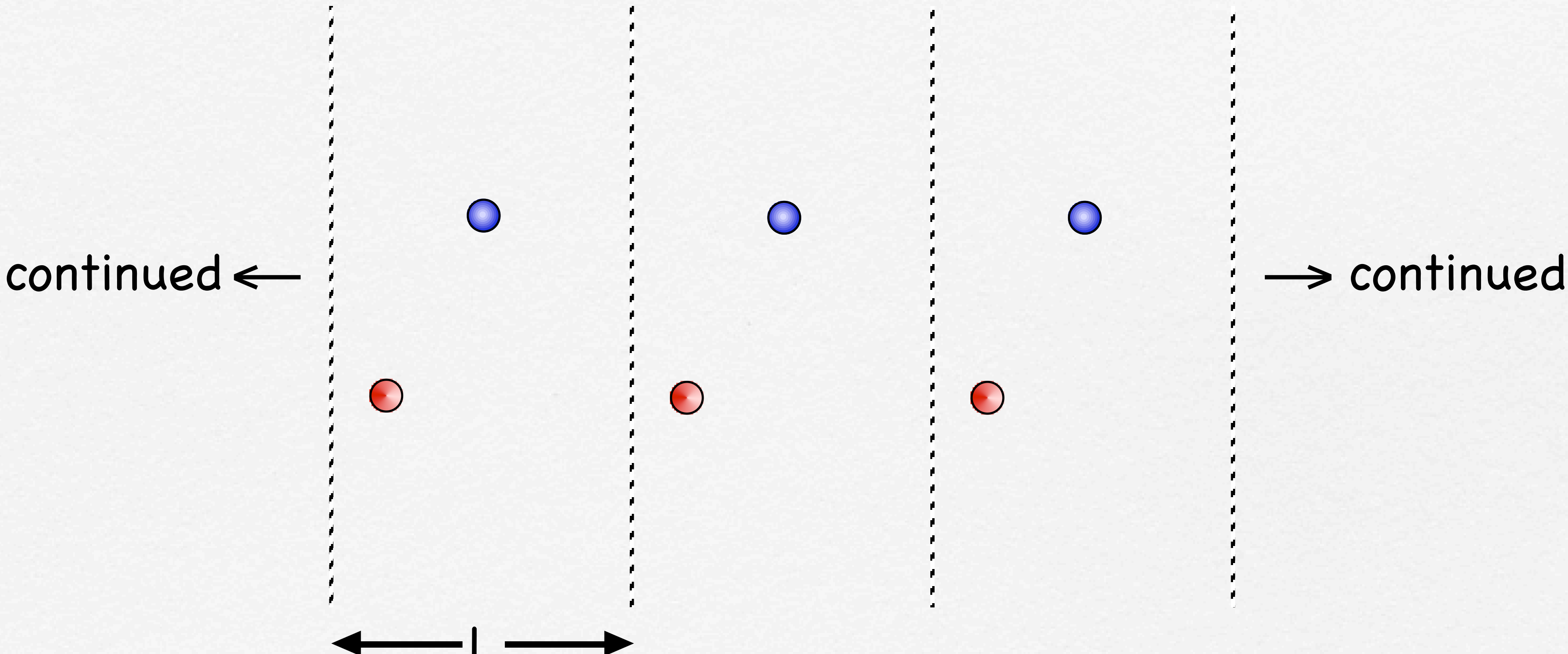
$$u - iv = \frac{1}{2Li}\sum_{\alpha=1}^{N}\Gamma_\alpha \cot\left[\frac{\pi(z - z_\alpha)}{L}\right]$$

Aleksandr Aleksandrovich Friedmann 1888 - 1925

Pelageia Yakovlena Polubarinova Kochina 1899-1999

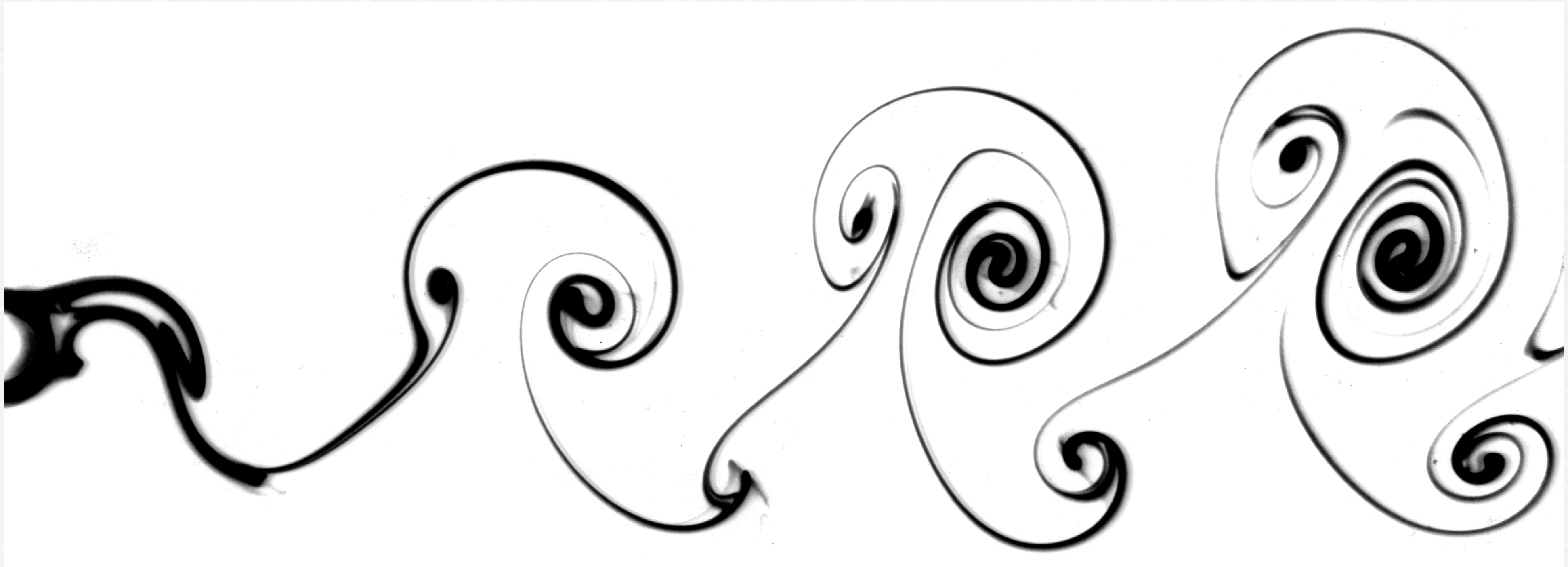


Kármán vortex street as a two-vortices-in-a-periodic-strip configuration

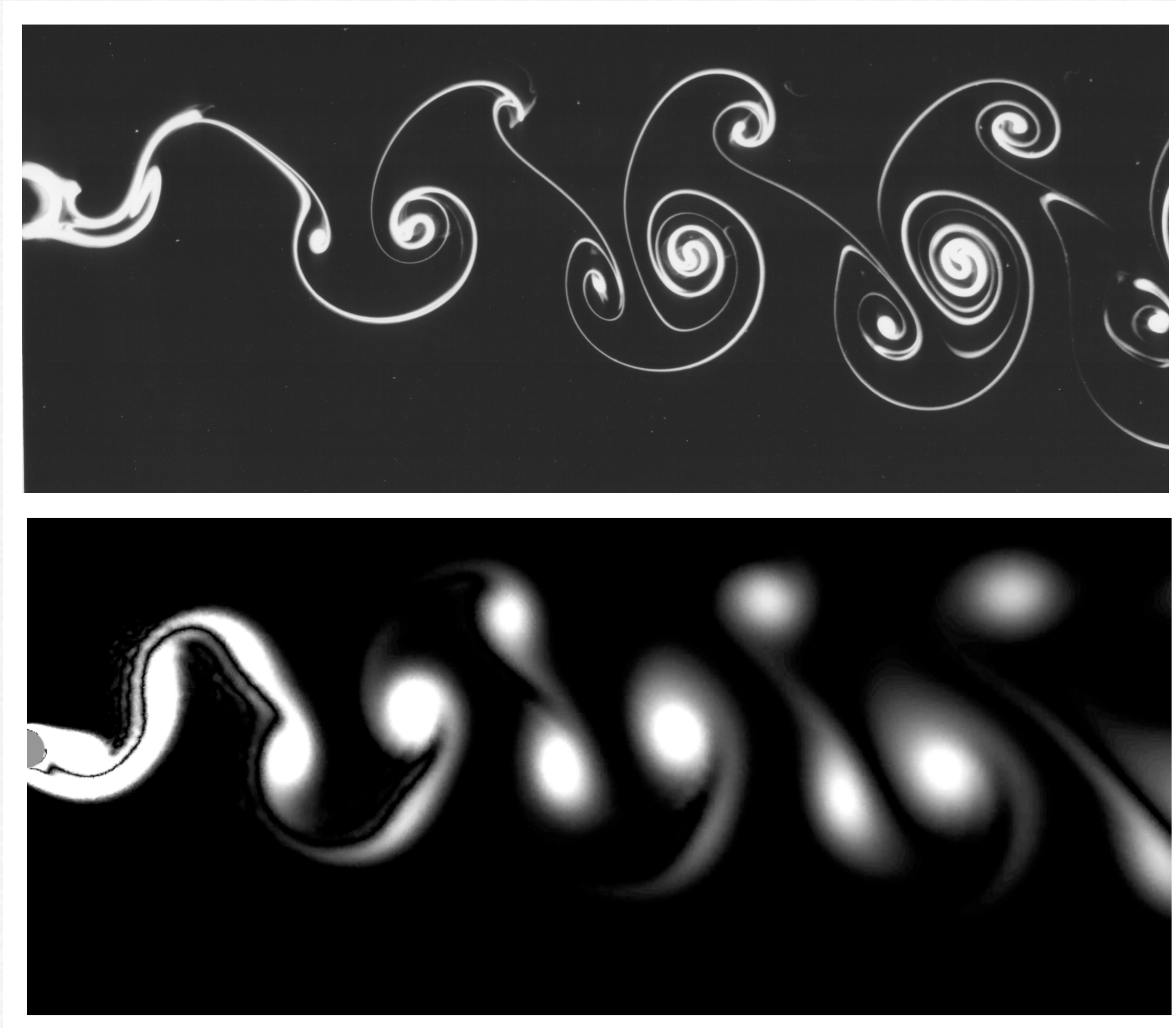


$$\dot{z}^*_\alpha = \frac{1}{2Li} \sum_{\beta=1}^{N}{}' \, \Gamma_\beta \cot\left[\frac{\pi}{L}\left(z_\alpha - z_\beta\right)\right]$$

Shedding from an oscillating cylinder can produce vortex wake patterns with three vortices (net circulation zero) per cycle



Re = 140 photo by C. H. K. Williamson



Comparison of flow visualization and 2D simulation (FE with vortex method to give shedding at cylinder)

From Ponta & Aref, 2006

Three vortices, total circulation = 0, in a periodic strip of width L

$$\Gamma_1 z_1 + \Gamma_2 z_2 + \Gamma_3 z_3 = X + iY$$

$$\Gamma_1 + \Gamma_2 + \Gamma_3 = 0$$

$$z_1 - z_2 = Z$$

$$\Xi = X + iY$$

$$z_2 - z_3 = \frac{\Gamma_1 Z - \Xi}{\Gamma_3}$$

Also:

$$\gamma = \frac{\Gamma_2}{\Gamma_3} + \frac{1}{2}$$

$$z_3 - z_1 = \frac{\Gamma_2 Z + \Xi}{\Gamma_3}$$

Aref & Stremler, 1996

$$\dot{z}_1^* = -i\left(\Gamma_2 C(z_1 - z_2) + \Gamma_3 C(z_1 - z_3)\right)$$

$$\dot{z}_2^* = -i\left(\Gamma_3 C(z_2 - z_3) + \Gamma_1 C(z_2 - z_1)\right)$$

$$\dot{z}_3^* = -i\left(\Gamma_1 C(z_3 - z_1) + \Gamma_2 C(z_3 - z_1)\right)$$

$$C(z) = \frac{\cot\left(\frac{\pi}{L} z\right)}{2L}$$

$$\gamma = \frac{\Gamma_2}{\Gamma_3} + \frac{1}{2}$$

$$Q = -\frac{X + iY}{\Gamma_3}$$

$$\dot{Z}^* = i\Gamma_3\left(C(Z) + C\left[Q - \left(\gamma + \tfrac{1}{2}\right)Z\right] - C\left[Q - \left(\gamma - \tfrac{1}{2}\right)Z\right]\right)$$

Aref & Stremler, 1996

$$\dot{Z}^* = i\Gamma_3\left(C(Z) + C\left[Q - \left(\gamma + \tfrac{1}{2}\right)Z\right] - C\left[Q - \left(\gamma - \tfrac{1}{2}\right)Z\right]\right)$$

This ODE can be interpreted* in terms of advection by three vortex rows:

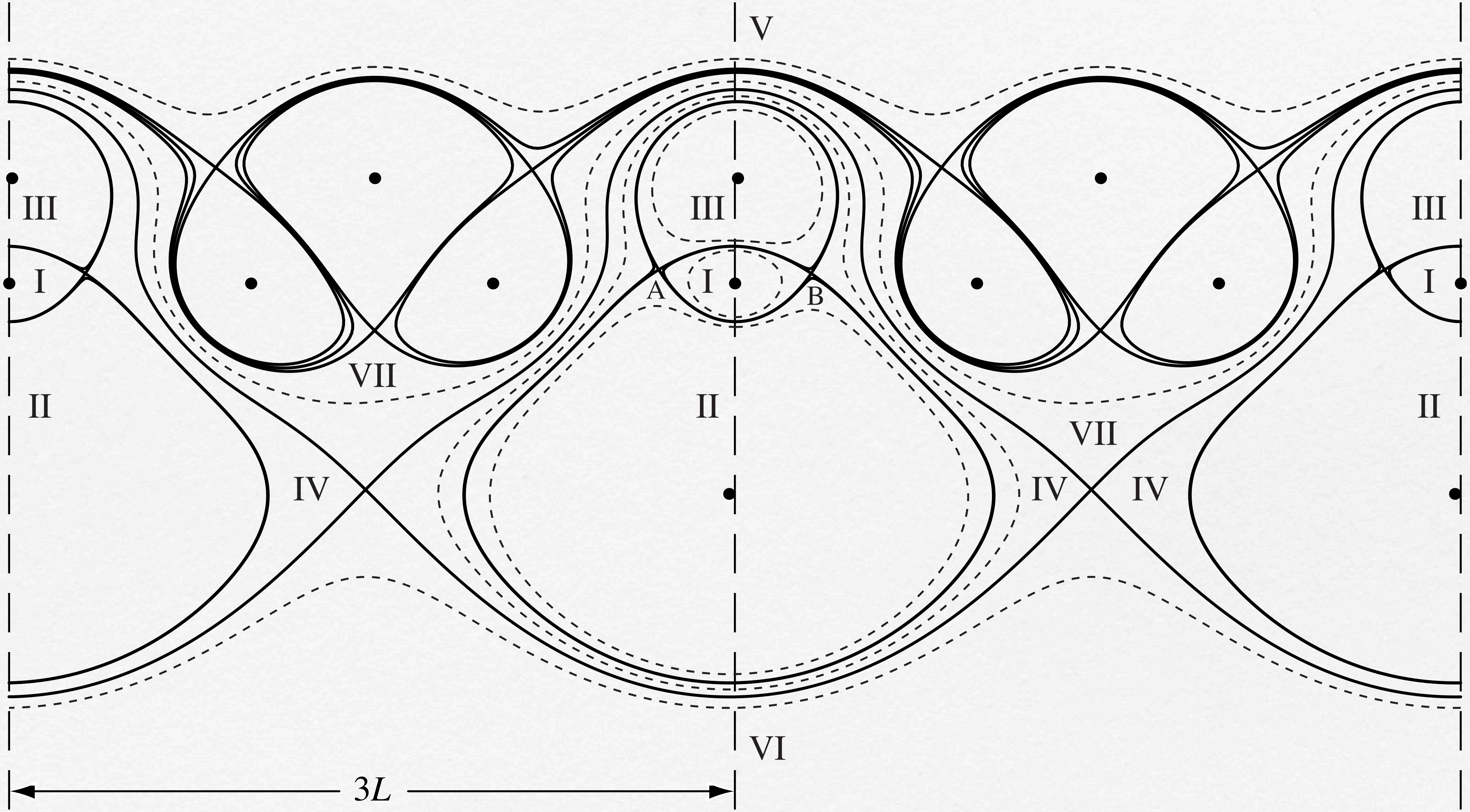
Circulation	Positions
Γ_3^{-1}	nL
Γ_2^{-1}	$\dfrac{-\Xi + nL\Gamma_3}{\Gamma_2}$
Γ_1^{-1}	$\dfrac{\Xi + nL\Gamma_3}{\Gamma_1}$

Note: If γ is rational, rows are commensurate!

$n = 0, \pm 1, \pm 2, \ldots$

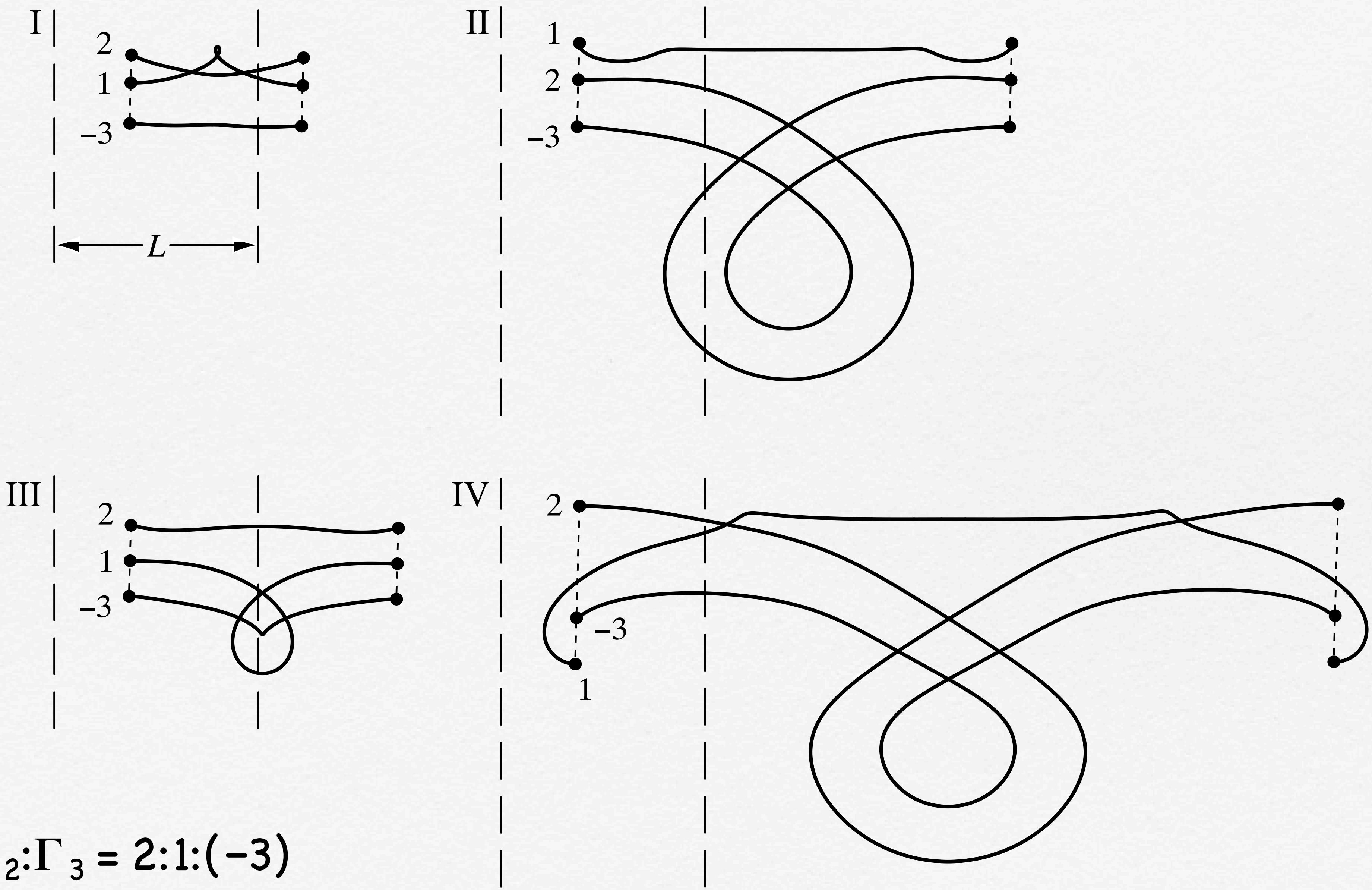
*) After some transformation

Aref & Stremler, 1996



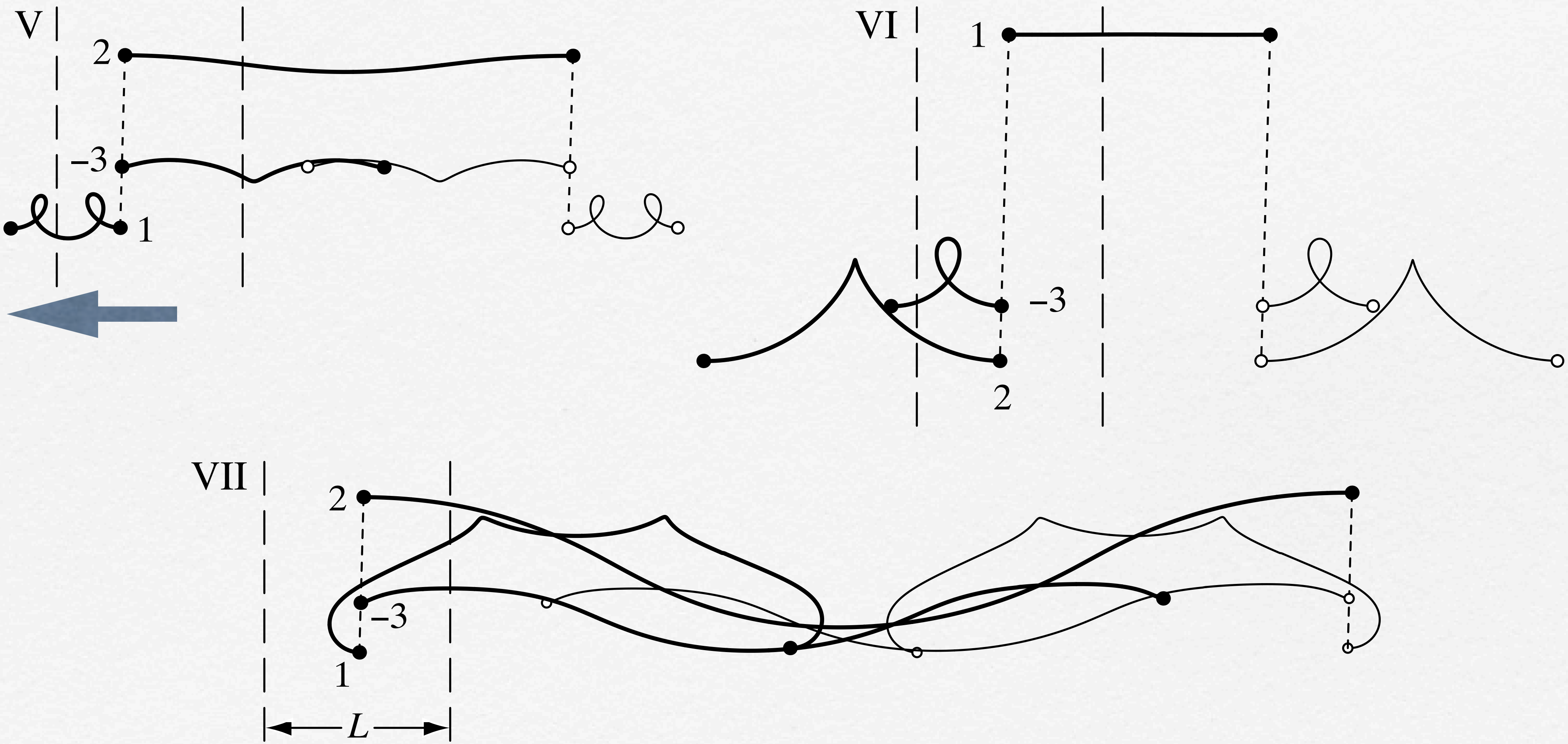
$\Gamma_1:\Gamma_2:\Gamma_3 = 2:1:(-3)$

Trajectories that correspond to infinite plane case:



$\Gamma_1 : \Gamma_2 : \Gamma_3 = 2 : 1 : (-3)$

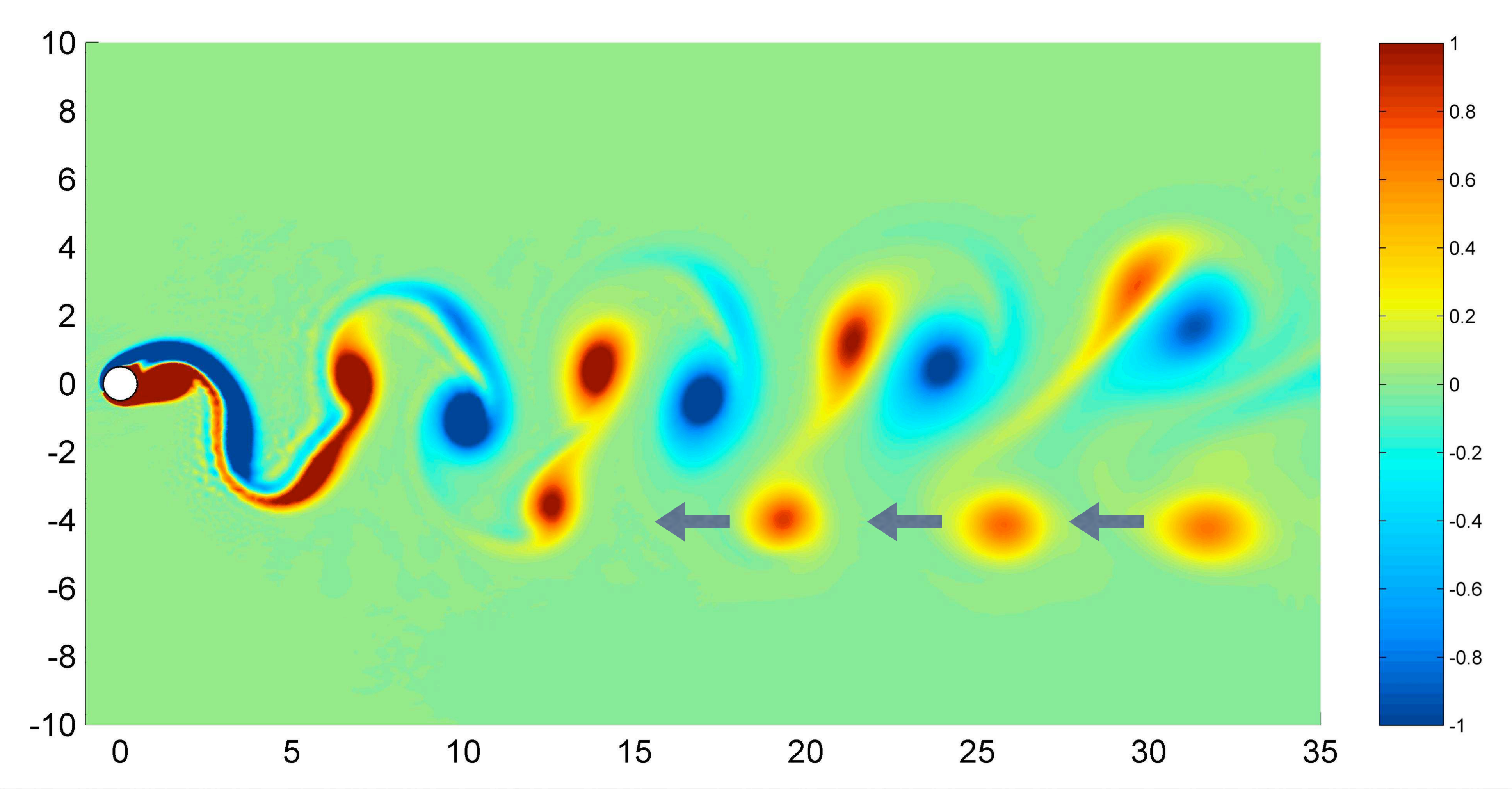
More “exotic” trajectories:



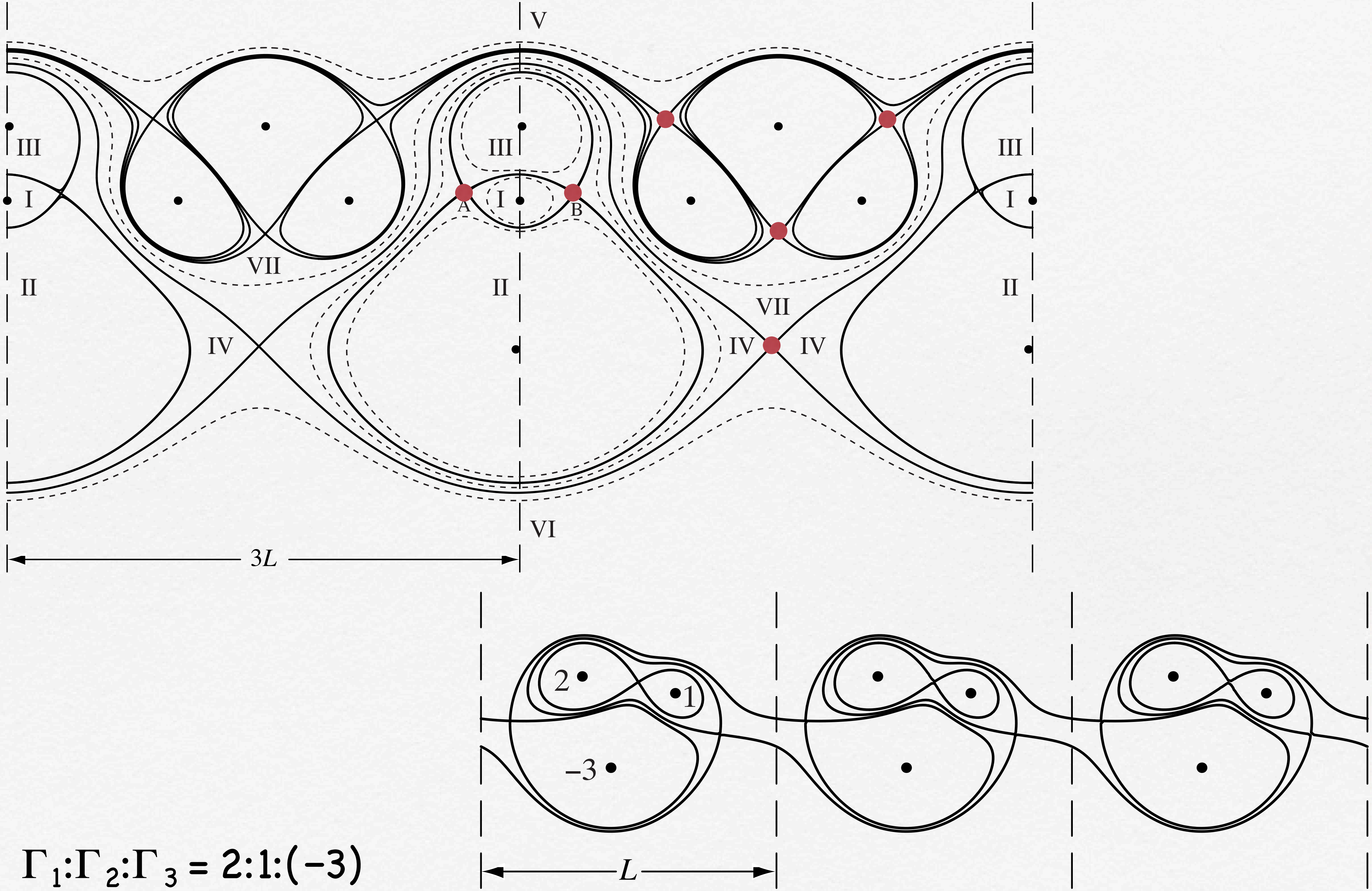
$\Gamma_1:\Gamma_2:\Gamma_3$ = 2:1:(−3)   Commensurate case – relative motion is periodic!

Simulation of three-vortex shedding from an oscillating cylinder

Parameters: Re = 140, A/d = 1, λ/d = 7.5

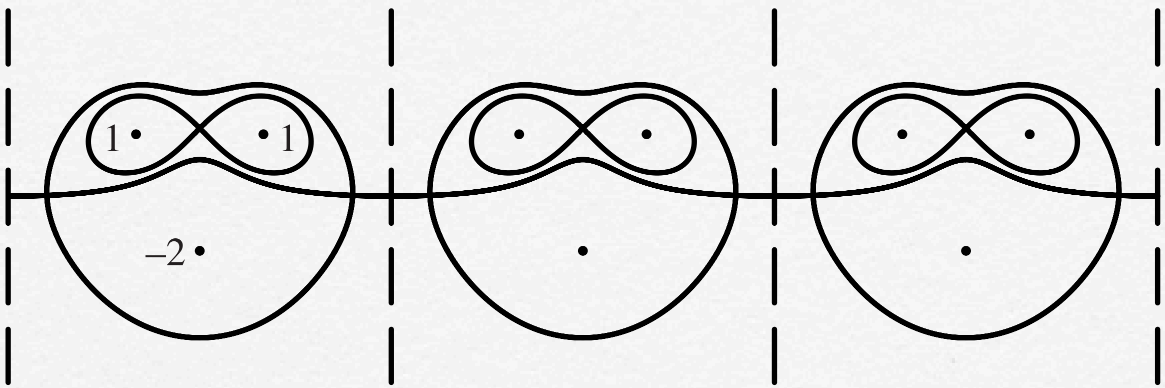


Fernando Ponta, 2003



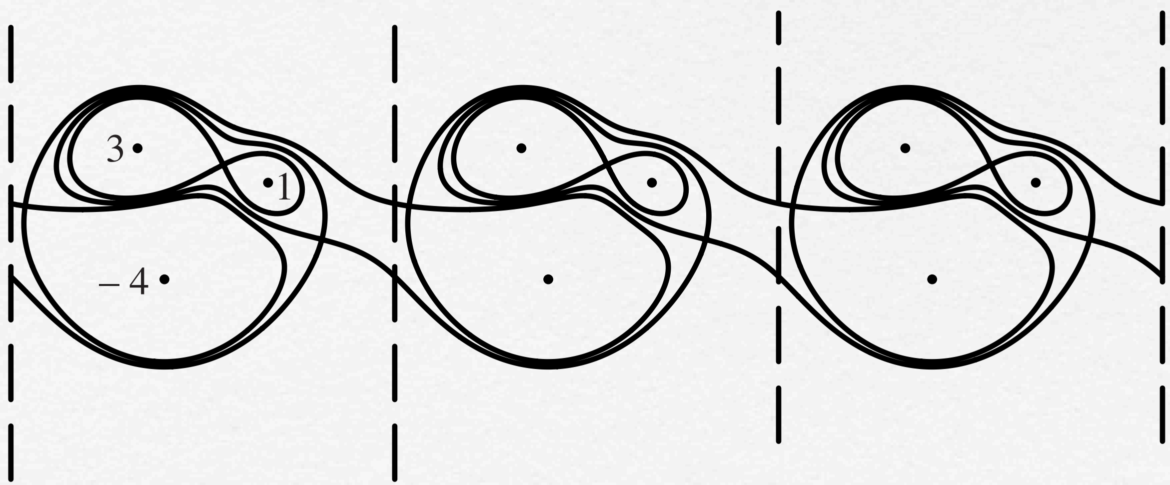
$\Gamma_1:\Gamma_2:\Gamma_3 = 2:1:(-3)$

Three-vortices-per-period relative equilibria can be found analytically (Stremler 2003)

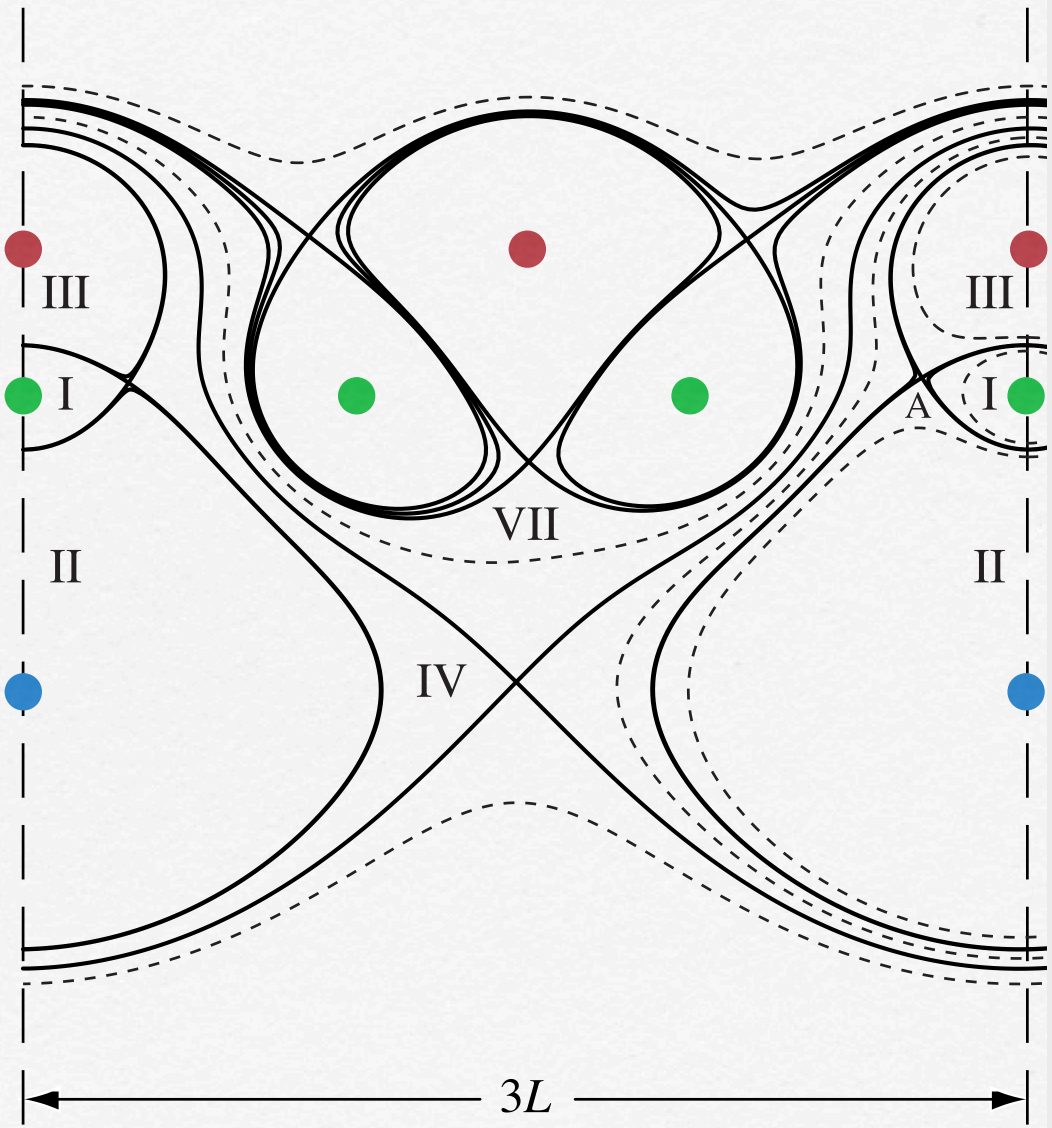


1:1:(−2)

3:1:(−4)



Other examples where translation is along the rows – oblique streets also in abundance!



$\Gamma_1 : \Gamma_2 : \Gamma_3 = 2:1:(-3)$

Bonus

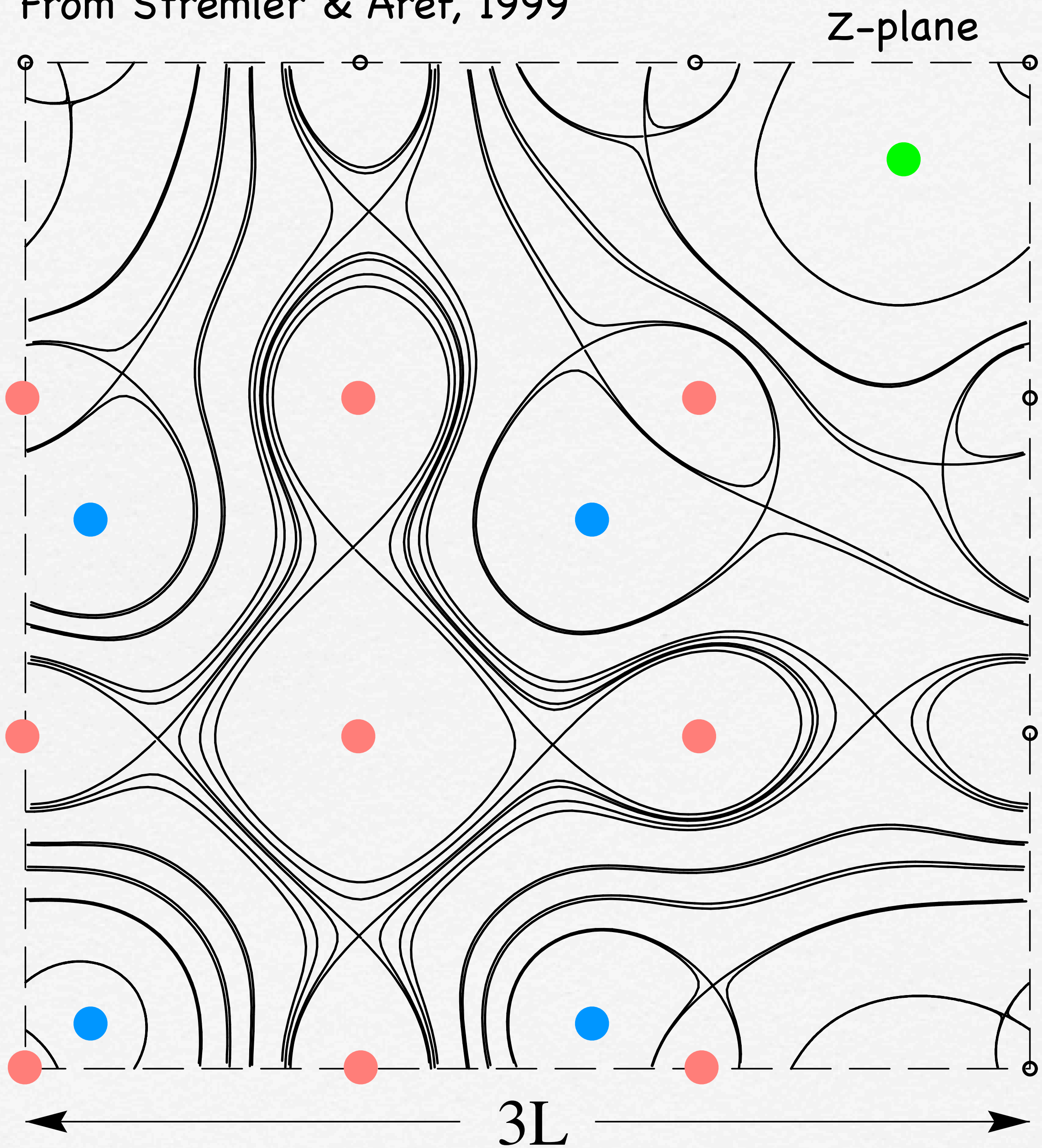
Advecting configurations are steady states!

Momentumless wakes?

Non-zero net circulation...

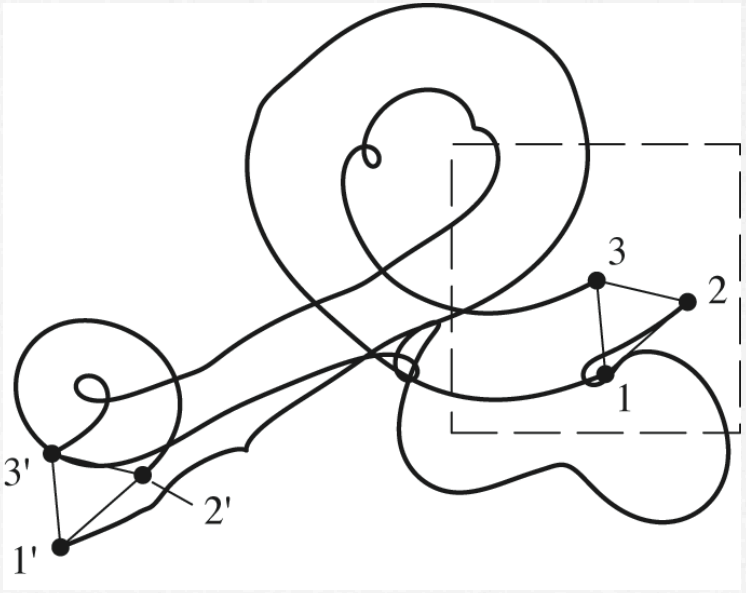
Periodic for rational ratio of strengths, but aperiodic otherwise!

From Stremler & Aref, 1999

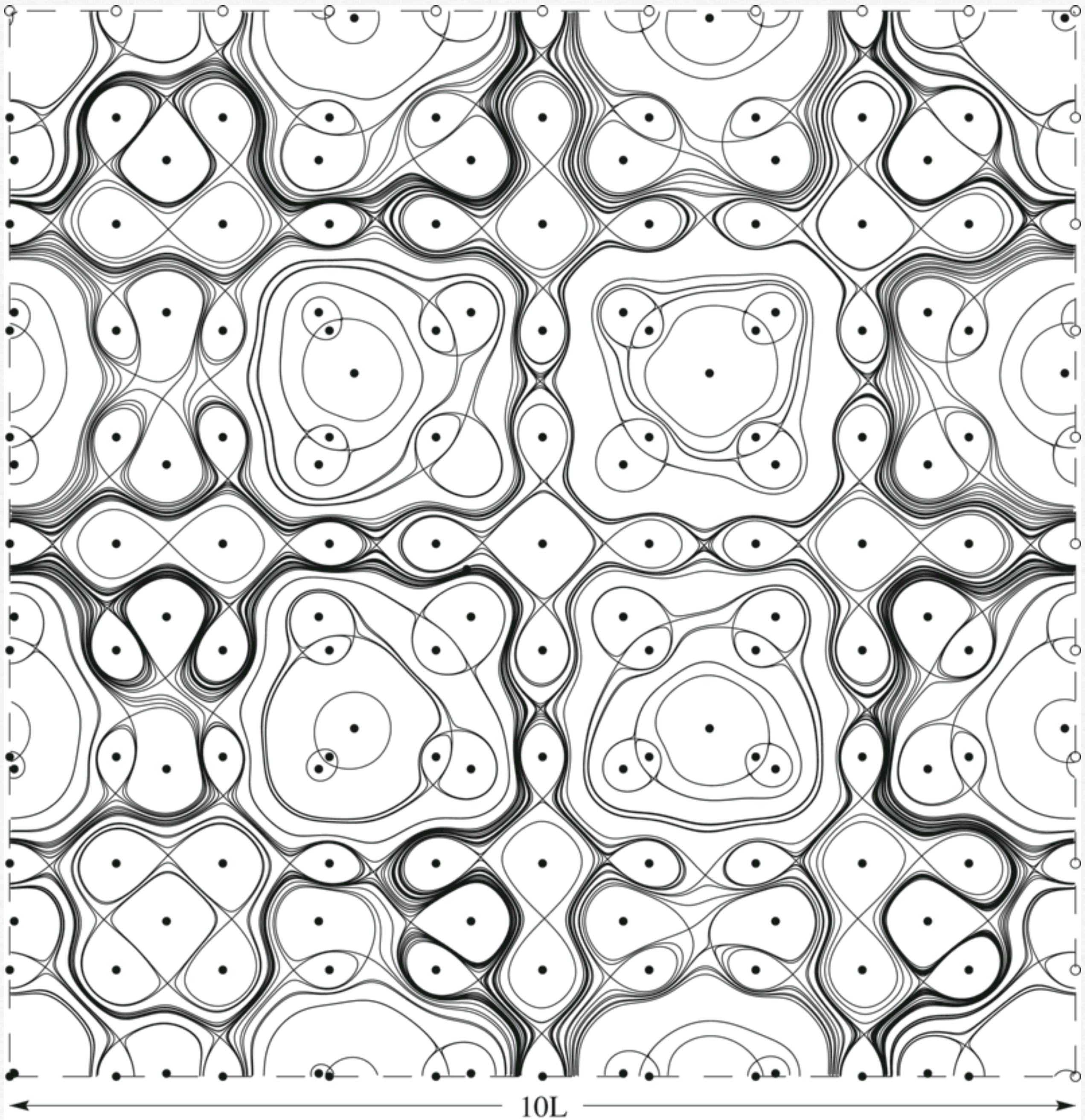


Advection problem for three vortices, $\Gamma_1:\Gamma_2:\Gamma_3 = 2:1:(-3)$

Advecting vortices of the three “families”



Sample trajectory corresponding to a “complex” regime



Advection problem for

$\Gamma_1 : \Gamma_2 : \Gamma_3 = 9:1:(-10)$

"Complex regimes" produce "maximal chaotic advection"

Advecting vortices are steady state in square of side 10L

From Stremler & Aref, 1999

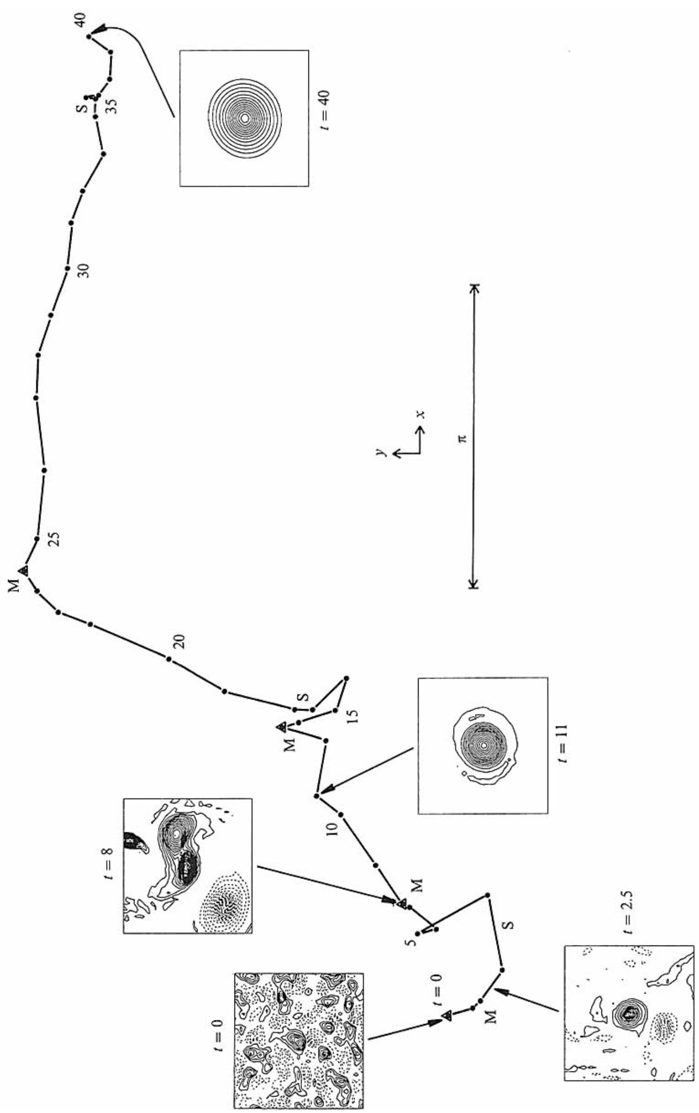
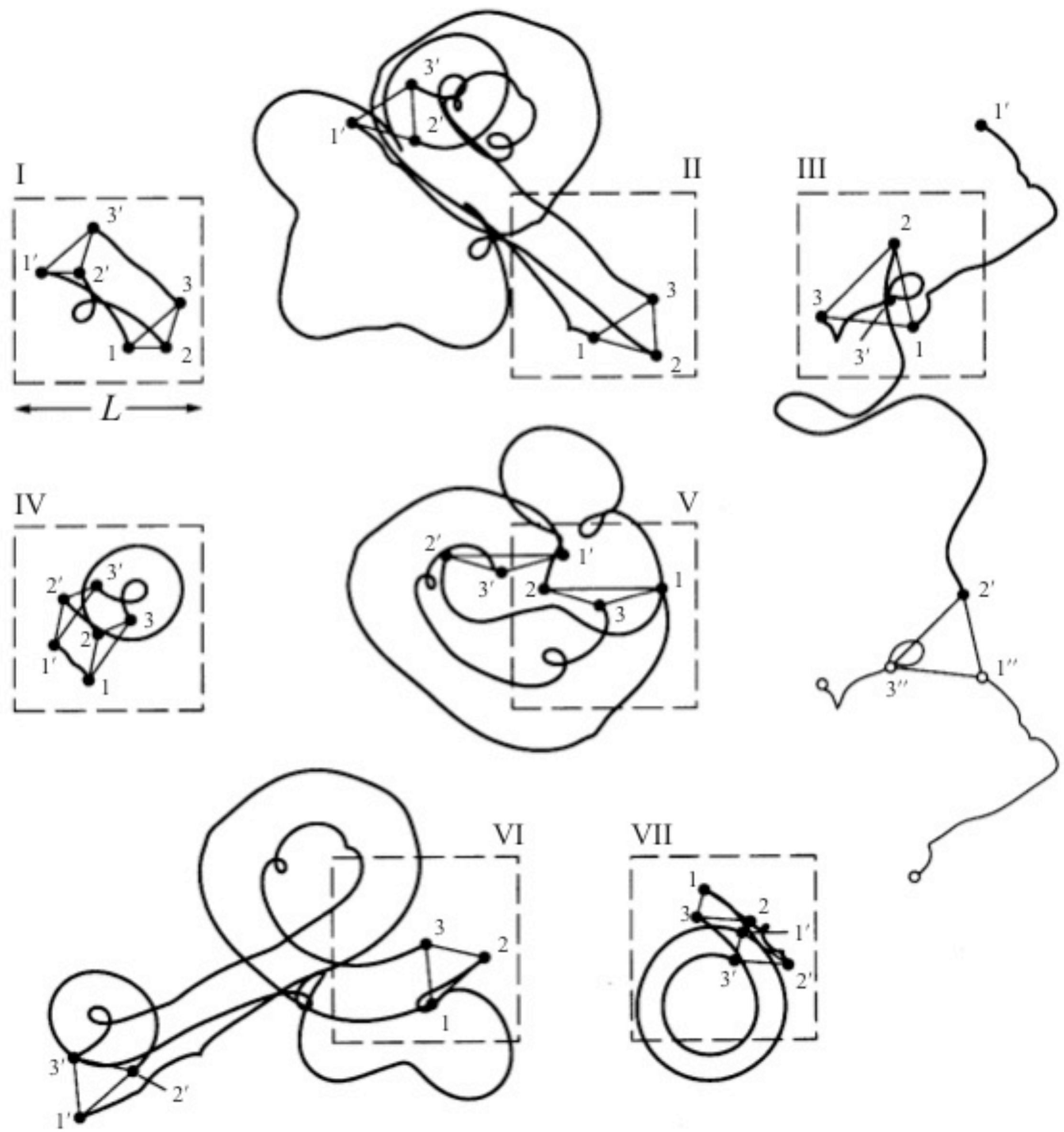


FIGURE 2. For caption see facing page.

M. A. Stremler and H. Aref



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The End